48, DEVLIN

"MATHEMATICS: SCIENCE OF PATTERNS"

Patterns of Truth

The logical operations used to combine propositions into more complex propositions are formally defined by showing how the truth or falsity of the compound is related to that of the individual components. This type of definition is most conveniently represented in a tabular form, as a so-called truth table.

The tables below, which capture certain 'patterns of truth', are labeled using the mathematical symbols commonly used nowadays: $\wedge$ to denote conjunction (*and*), $\vee$ to denote disjunction (*or*), and $\neg$ to denote negation (*not*). These tables provide the formal definitions of these logical operations. This use of truth tables motivates the use of the term 'truth *value*' to denote the truth (value = T) or falsity (value = F) of a particular proposition.

Reading along a row, each table indicates the truth value of the compound that arises from the truth values of the components.

p	q	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

p	q	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

p	q	$p \to q$
T	T	T
T	F	F
F	T	T
F	F	T

p	$\neg p$
T	F
F	T

Since the various logical operations are defined purely in terms of their truth patterns, if two compound propositions have truth tables that are row-by-row identical, then the two compounds are, to all intents and purposes, equal. By computing truth tables, the following 'laws of logical algebra' can be obtained:

$$p \wedge q = q \wedge p \qquad p \vee q = q \vee p$$

$$p \wedge (q \wedge r) = (p \wedge q) \wedge r$$

$$p \vee (q \vee r) = (p \vee q) \vee r$$

$$p \wedge (q \vee r) = (p \wedge q) \vee (p \wedge r)$$

$$p \vee (q \wedge r) = (p \vee q) \wedge (p \vee r)$$

$$p \wedge \mathrm{T} = p \qquad p \vee \mathrm{T} = \mathrm{T}$$

$$p \wedge \mathrm{F} = \mathrm{F} \qquad p \vee \mathrm{F} = p$$

$$\neg(p \wedge q) = (\neg p) \vee (\neg q)$$

$$\neg(p \vee q) = (\neg p) \wedge (\neg q)$$

$$\neg\neg p = p$$

$$p \to q = (\neg p) \vee q$$

In these equations, T denotes any true proposition, such as $1 = 1$, and F denotes any false proposition, such as $0 = 1$.

Apart from the last one, these equations exhibit some similarities to the familiar laws of arithmetic. A much stronger connection unites the above patterns and those in Boole's logic. If $p \wedge q$ is taken to correspond to Boole's pq, $p \vee q$ to Boole's $p + q$, and $\neg p$ to Boole's $1 - p$, and if T, F are taken to correspond to Boole's 1, 0, respectively, then all of the above identities (apart from the last one) hold for Boole's logic.

In all of these identities, the 'equality' is not genuine equality. All it means is that the two propositions concerned have the same truth table. In particular, with this meaning of 'equals' the following is true:

7 is a prime = *the angles of a triangle sum to* 180°.

Because of this very special meaning of 'equality', mathematicians generally use a different symbol, writing $\equiv$ or $\leftrightarrow$ instead of $=$ in such identities.