Math 105 *Problem of the Day #2*

Name Key

Triangular Numbers

The numbers 1, 3, 6,... are called the first three *triangular numbers* since they may be represented by triangular patterns of dots.

(a) Draw pictures for the first few triangular numbers here...

1 3 6 10 15

(b) Make a table of the first 10 triangular numbers and, for any n, give a formula for the n-th triangular number. Call the n^{th} triangular number T_n. To prove your formula, try to give a picture with dots which illustrates your result. (Notice that what you have actually found is a formula for the sum of the first n natural numbers.)

$$T_n = \frac{n(n+1)}{2}$$

n	T_n	n	T_n
1	1	7	28
2	3	8	36
3	6	9	45
4	10	10	55
5	15	11	66
6	21	12	78

(c) What is the sum of any two consecutive triangular numbers? That is find a formula for $T_n + T_{n+1}$. Prove your answer is correct, using algebra and/or your result from the previous question in (b). Now draw a picture with dots to illustrate this result.

$$T_{n+1} = \frac{(n+1)(n+2)}{2}$$

$$T_n = \frac{n(n+1)}{2}$$

$$T_{n+1} + T_n = \frac{(n+1)(n+2)}{2} + \frac{n(n+1)}{2}$$

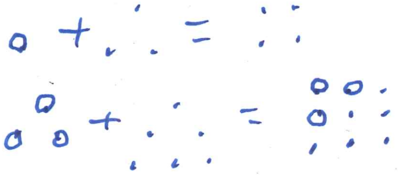
$$= \frac{(n+1)}{2}(n+2+n)$$

$$= (n+1)\frac{(2n+2)}{2}$$

$$= (n+1)^2$$

$$= S_{n+1}$$

$T_1 + T_2 = S_2 = 2^2$



$T_2 + T_3 = S_3 = 3^2$

(d) Prove that if T_n is a triangular number, then so is $9T_n + 1$.

$$T_n = \frac{n(n+1)}{2}$$

$$9T_n + 1 = \frac{9n(n+1)}{2} + 1$$

$$= \frac{9n(n+1)}{2} + \frac{2}{2}$$

$$= \frac{9n(n+1)+2}{2}$$

$$= \frac{9n^2+9n+2}{2} = \frac{(3n+1)(3n+2)}{2}$$

$$= \frac{(3n+1)((3n+1)+1)}{2} = T_{3n+1}$$

(e) Explain why each number in the following sequence is a triangular number:

$$1, 10, 91, 820, ...$$

HINT: you can think of the above sequence as...

$$1, 1+9, 1+9+81, ..., 1+9+9^2+...+9^k, ...$$

(OMITTED from Spring 2015)