Triangular Numbers

The numbers 1, 3, 6, ... are called the first three triangular numbers since they may be represented by triangular patterns of dots.

(a) Draw pictures for the first few triangular numbers here...

1  3  6  10  15

(b) Make a table of the first 10 triangular numbers and, for any $n$, give a formula for the $n$-th triangular number. Call the $n$-th triangular number $T_n$. To prove your formula, try to give a picture with dots which illustrates your result. (Notice that what you have actually found is a formula for the sum of the first $n$ natural numbers.)

\[
T_n = \frac{n(n+1)}{2}
\]

\[
\begin{array}{c|c|c}
T_n & n & T_n \\
--- & --- & --- \\
1 & 1 & 1 \\
2 & 3 & 3 \\
3 & 6 & 6 \\
4 & 10 & 10 \\
5 & 15 & 15 \\
6 & 21 & 21 \\
7 & 28 & 28 \\
8 & 36 & 36 \\
9 & 45 & 45 \\
10 & 55 & 55 \\
\end{array}
\]

(c) What is the sum of any two consecutive triangular numbers? That is find a formula for $T_n + T_{n+1}$. Prove your answer is correct, using algebra and/or your result from the previous question in (b). Now draw a picture with dots to illustrate this result.

\[
T_{n+1} = \frac{(n+1)(n+2)}{2}
\]

\[
T_n + T_{n+1} = \frac{n(n+1)}{2} + \frac{(n+1)(n+2)}{2}
\]

\[
= \frac{(n+1)(n+2+n)}{2}
\]

\[
= \frac{2(n+1)(2n+2)}{2}
\]

\[
= (n+1)^2
\]

\[
= S_{n+1}
\]
(d) Prove that if \( T_n \) is a triangular number, then so is \( 9T_n + 1 \).

\[
T_n = \frac{n(n+1)}{2}
\]

\[
9T_n + 1 = \frac{9n(n+1) + 1}{2}
\]

\[
= \frac{9n(n+1) + 2}{2}
\]

\[
= \frac{9n(n+1) + 2}{2}
\]

\[
= \frac{9n^2 + 9n + 2}{2} = \frac{(3n+1)(3n+2)}{2}
\]

\[
= \frac{(3n+1)(3n+1+1)}{2} = T_{3n+1}
\]

(e) Explain why each number in the following sequence is a triangular number:

\[1, 10, 91, 820, \ldots\]

**HINT:** you can think of the above sequence as...

\[1, 1 + 9, 1 + 9 + 81, \ldots, 1 + 9 + 9^2 + \ldots + 9^k, \ldots\]

(COMMITTED FROM SPRING 2015)