Complex Analysis

Math 312 Spring 2016 | Fowler 309 MWF 11:45am-12:40pm

 | http://sites.oxy.edu/ron/math/312/16/

Class 8: **Friday February 5**

TITLE Power Functions, The Reciprocal Function and Point at Infinity
CURRENT READING Zill & Shanahan, Section 2.4 & 2.5
HOMEWORK SET #3 (DUE WED FEB 10)
Zill & Shanahan, §2.1: #3, 8, 14, 20, 36, **27***; §2.2: 7, 11, 12, 22, **27***
Zill & Shanahan, §2.3: 9, 18, 19, 34, **29***.
Zill & Shanahan, §2.4: 23, 25, 31 **47***; §2.5: 4, 16, 22, **25***.

SUMMARY
We shall consider two important functions, the Reciprocal Function $f(z) = \frac{1}{z}$ and the Principal Square root function and introduce the idea of the (infamous) "Point at Infinity."

Point at Infinity

When dealing with real numbers we often speak of two different concepts, denoted $-\infty$ and $+\infty$. These symbols are our representation of the idea that a real number can grow without bound in a positive direction or a negative direction.

However, in the complex plane, infinity is represented as one particular point in the Argand plane. (Recall, the relational operators $<$ or $>$ are not defined for complex numbers. We have no way of determining whether a complex number is "positive" or "negative" or greater or lesser than any number.)

The idea of a complex number growing without bound is and denote as ∞ and represented in the complex plane as the **point at infinity**. We rename the Argand plane the **extended** z **plane** or the **extended complex plane** when we include ∞. Points in the extended complex plane "near" the point at infinity are points in the extended complex plane with extremely large values of their modulus $|z|$.

The point at infinity can be considered to be the image of the origin $z = 0$ under the mapping $w = 1/z$.

Reciprocal Function

The function $w = \frac{1}{z}$, known as the reciprocal function can be defined as

$$f(z) = \begin{cases} \dfrac{1}{z}, & \text{if } z \neq 0 \text{ or } \infty \\ \infty, & \text{if } z = 0 \\ 0, & \text{if } z = \infty \end{cases}$$

Reciprocal Function as a Mapping

The reciprocal function can be thought of as the composition of two mappings: "inversion in the unit circle" and conjugation (i.e. reflection about the real axis).
Let $z = re^{i\theta}$. Under the mapping $w = 1/z$,

$$w = \frac{1}{re^{i\theta}} = \frac{1}{r}e^{-i\theta} = \overline{\frac{1}{r}e^{i\theta}}$$

Notes about the reciprocal mapping

- The reciprocal function only maps circles and lines to either a circle or a line.
- When thinking about the mapping under the reciprocal function, everything that is inside the unit circle $|z| = 1$ gets mapped to everything outside $|w| = 1$ and then reflected about the real axis.
- If the pre-image includes the origin, then the image under the reciprocal mapping must include the point at infinity, (i.e. **it must be a line**)
- if the pre-image does NOT include the origin, then the image under the reciprocal mapping must NOT include the pint at infinity, (i.e. **it must be a circle**)
- If the pre-image is a line (i.e. it includes the point at infinity), then the image under the reciprocal mapping must include the origin, (i.e. **it could be a line OR a circle**)

Exercise 1

Show that the image of the circle $|z-1| = 1$ under the mapping $w = 1/z$ is the line $\text{Re } z = \frac{1}{2}$

$\boxed{\text{EXAMPLE 1}}$

Let's show that the image of the line $\mathrm{Re}(z) = 1$ under the mapping $w = 1/z$ is the circle $\left|w - \frac{1}{2}\right| = \frac{1}{2}$

Reciprocal Function Maps Lines To Circles (and Circles to Circles)

The reciprocal function on the extended complex plane maps

(i) the vertical line $x = k$ with $k \neq 0$ to the circle $\left|w - \dfrac{1}{2k}\right| = \left|\dfrac{1}{2k}\right|$

(ii) the horizontal line $y = k$ with $k \neq 0$ to the circle $\left|w + i\dfrac{1}{2k}\right| = \left|\dfrac{1}{2k}\right|$

(iii) the circle $|z| = k$ with $k \neq 0$ to the circle $|w| = \left|\dfrac{1}{k}\right|$

Principal Square Root Function

The principal square root function is the function $w = z^{1/2}$ or $w = \sqrt{z}$ which is defined as $|z|^{1/2}e^{\frac{i\mathrm{Arg}(z)}{2}}$ or $|z|^{1/2}\exp\left(\dfrac{i\mathrm{Arg}(z)}{2}\right)$.

This expression is the single valued version of the formula $z^{1/2} = |z|^{1/2}\exp\left(\dfrac{i\mathrm{Arg}(z) + 2k\pi i}{2}\right)$ where $k = 0$ and $k = 1$. The **principal value** just takes the $k = 0$ value.

$\boxed{\textbf{Exercise}}$

Find the principal square root of the following complex numbers:

(a) 4

(b) $-2i$

(c) $-1 + i$