**Section 1:** Metric spaces; open and closed sets; limit points; interior, closure, boundary; continuity.

### Definition 1. A **metric space** \( M \) consists of a set \( X \) and a **distance function** \( d : X \times X \to [0, \infty) \) such that \( \forall x, y, z \in X \):

1. \( d(x, y) = 0 \) iff \( x = y \);
2. \( d(x, y) = d(y, x) \) (\( d \) is symmetric);
3. \( d(x, z) \leq d(x, y) + d(y, z) \) (triangle inequality).

#### Example 1. \( \mathbb{R} \) with the **Euclidean metric** (the “standard” metric):
\( X = \mathbb{R} \), \( d(x, y) = |x - y| \). Why is this a metric space? If we replace \( d \) with \( d(x, y) = x - y \), will we still have a metric space?

#### Example 2. \( \mathbb{R} \) with the **discrete metric**, denoted \( \mathbb{R}_d \):
\( X = \mathbb{R} \), \( d(x, y) = \left\{ \begin{array}{ll} 0 & \text{if } x = y \\ 1 & \text{if } x \neq y \end{array} \right. \). Why is this a metric space? How about \( d(x, y) = 0 \ \forall x, y \)?

#### Example 3. \( \mathbb{R}^n \) with the **Euclidean metric**:
\( X = \mathbb{R} \times \cdots \times \mathbb{R} \) (\( n \) times), for \( x = (x_1, \ldots, x_n) \), \( y = (y_1, \ldots, y_n) \), \( d(x, y) = \sqrt{(x_1 - y_1)^2 + \cdots + (x_n - y_n)^2} \). Why is this a metric space? Conditions 1 and 2 of the definition (above) are clearly satisfied. Condition 3 is the well-known triangle inequality (skip proof).

#### Example 4. \( \mathbb{R}^2 \) with the **taxicab metric**:
\( X = \mathbb{R}^2 \), for \( a = (a_1, a_2) \), \( b = (b_1, b_2) \), \( d(a, b) = |a_1 - b_1| + |a_2 - b_2| \). Why is this a metric space? (HW)

#### Note.

1. Unless stated otherwise, whenever we refer to \( \mathbb{R} \) as a metric space without stating what the distance function \( d \) is, we mean “\( \mathbb{R} \) with the Euclidean metric.”

2. For a metric space \( M = (X, d) \), \( X \) is called the **underlying set**. We will often abuse notation and write \( M \) instead of \( X \), or vice versa; for example, we may write \( x \in M \) instead of \( x \in X \); or we may refer to \( X \) as a metric space, when it’s really \( M = (X, d) \) that’s a metric space.

### Definition 2. Given a metric space \( M \), a point \( x \in M \), and a real number \( r \geq 0 \), the **ball** of radius \( r \) around \( x \) is defined as
\[ B_r(x) = \{ y \in M \mid d(x, y) < r \} \]

#### Example 5. In \( \mathbb{R} \) with the Euclidean metric, \( B_2(1) = ? \)

#### Example 6. In \( \mathbb{R}^2 \) with the Euclidean metric, what does \( B_2(1, 2) \) look like? (Strictly speaking, we should write \( B_2((1, 2)) \); but too many parentheses can make it difficult to read, so we slightly abuse notation and write only one set of parentheses.) How about \( B_2(1, 2) \subset \mathbb{R}^3 \), what does it look like?

#### Example 7. In \( \mathbb{R}_d \), what is \( B_3(8) \)? What is \( B_{0.5}(8) \)?

#### Example 8. In \( \mathbb{R}^2 \) with the taxicab metric, what does \( B_1(0, 0) \) look like?

#### Example 9. Is there a metric on \( \mathbb{R}^2 \) for which \( B_1(0, 0) = (-1, 1) \times (-1, 1) \)?

### Definition 3. A subset \( A \) of a metric space \( M \) is said to be **open** in \( M \) iff \( \forall x \in A \), \( \exists r > 0 \) such that \( B_r(x) \subset A \).

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1. The open interval from \(-1\) to \(3\): \((-1, 3)\).
2. \( B_3(8) = \mathbb{R} ; B_{0.5}(8) = \{8\} \).
3. \( d(a, b) = \max\{ |a_1 - b_1|, |a_2 - b_2| \} \).
Example 10. The interval $(-1, 1]$ is not open in $\mathbb{R}$. Why?  

Example 11. The interval $(-1, 1)$ is an open subset of $\mathbb{R}$. Why?

Proof: Given an arbitrary $x \in (-1, 1)$, let $r = \min\{d(x, 1), d(x, -1)\}$. Then, we prove as follows that $B_r(x) \subset (-1, 1)$. Let $y \in B_r(x)$; we’ll show $y \in (-1, 1)$. We will do so by showing that $d(0, y) < 1$. By definition of $B_r(x)$, $d(x, y) < r$; so $d(x, y) < \min\{d(x, 1), d(x, -1)\}$; so $d(x, y) < d(x, 1)$ and $d(x, y) < d(x, -1)$. By the triangle inequality, $d(0, y) \leq d(0, x) + d(x, y)$. So, $d(0, y) < d(0, x) + d(x, 1)$ and $d(0, y) < d(0, x) + d(x, -1)$. If $x \geq 0$, then the right hand side of the first inequality equals 1. If $x < 0$, then the left hand side of the second inequality equals 1. So either way, $d(0, y) < 1$, as desired. We showed that for every $x \in (-1, 1)$, there is a positive $r$ such that $B_r(x) \subset (-1, 1)$. So by the definition of open, $(-1, 1)$ is an open subset of $\mathbb{R}$. 

Example 12. Is the interval $(2, \infty)$ open in $\mathbb{R}$? Yes. Why?

Definition 4. Let $A$ be a subset of a metric space $M$. The complement of $A$ is $A^c = M - A$. $A$ is said to be closed in $M$ iff its complement $A^c$ is open in $M$. 

Example 13. $(-\infty, -1] \cup [1, \infty)$ is closed in $\mathbb{R}$. Why? 

Example 14. Is $(-\infty, -1]$ closed in $\mathbb{R}$? 

Example 15. Is $[-1, 1]$ closed in $\mathbb{R}$? 

Example 16. $(-1, 1)$ is neither open nor closed in $\mathbb{R}$. Why? 

Example 17. $\mathbb{R}$ is open in $\mathbb{R}$. Why? $\phi$ is open in $\mathbb{R}$. Why? 

Example 18. $\mathbb{R}$ is closed in $\mathbb{R}$. $\phi$ is closed in $\mathbb{R}$. Why? 

Example 19. Is the $x$-axis open or closed or neither in $\mathbb{R}^2$? 

Example 20. Find an open set in $\mathbb{R}_d$. Find a closed set in $\mathbb{R}_d$. (Quote from Munkres’s book, Topology: Q: “What’s the difference between a door and a set?” A: “A door is always either open or closed.”)

For emphasis, $B_r(x)$ is sometimes called the open ball of radius $r$ around $x$. In contrast, we have: 

Definition 5. The closed ball of radius $r$ around $x$ is defined as 

$$B_r(x) = \{y \in M \mid d(x, y) \leq r\}$$

Example 21. Draw the open and closed balls of radius 5 around the point 2 in $\mathbb{R}$. Draw the open and closed balls of radius 5 around the point $(2, 5)$ in $\mathbb{R}^2$. 

Definition 6. Let $A$ be a subset of a metric space $M$. A point $x \in M$ is said to be a limit point of $A$ iff every ball around $x$ contains a point of $A$ other than $x$. 

(Synonyms of limit point: cluster point; accumulation point.) 

Example 22. Let $M = \mathbb{R}$, $A = [0, 2]$. Which of the points $x = 0, 1, 2, 3$ are limit points of $A$? Why? 

What if $A = [0, 1] \cup \{2\}$? 

(Equivalent definition of limit point: $x$ is a limit point of $A$ iff $\forall \epsilon > 0, \exists y \in A - \{x\}$ such that $d(x, y) < \epsilon$.) 

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4Because there is no positive $r$ for which $B_r(1) \subset (-1, 1]$.
5Yes. Why?
6Yes. Why?
7Closed. Why?
8Each of $\mathbb{R}_d$ and $\phi$ is both open and closed.
90, 1 and 2.
100 and 1.
Theorem 1. A subset $A$ of a metric space $M$ is closed iff it contains all its limit points.

Proof. “$\Rightarrow$” : Suppose $A$ is closed. Then, by definition, $A^c$ is open. Let $x$ be a limit point of $A$. We want to show $x \in A$. By definition of limit point, every open ball around $x$ intersects $A - \{x\}$; therefore no open ball around $x$ is entirely contained in $A^c$. This implies $x \notin A^c$, since if $x$ were in $A^c$, then there would be an open ball around $x$ contained entirely in $A^c$ (since $A^c$ is open). Finally, since $x \notin A^c$, $x$ must be in $A$, as desired.

“$\Leftarrow$” : (Do yourself!)

Definition 7. Given a subset $A$ of a metric space $M$, its interior $A^\circ$ is defined as the set of all points $x \in A$ such that some open ball around $x$ is a subset of $A$. ($A^\circ$ is also written as Int $A$ or int($A$).)

Example 23. (a) What is the interior of $[2, 5) \subset \mathbb{R}$? 11
(b) What is the interior of $(2, 5) \subset \mathbb{R}$? 12
(c) What is the interior of the closed ball of radius 2 around the origin in $\mathbb{R}^2$? 13

Definition 8. Given a subset $A$ of a metric space $M$, its closure $\overline{A}$ is defined as $A$ union the set of all limit points of $A$. The boundary of $A$ is defined as $\partial A = \overline{A} - A^\circ$.

Example 24. (a) What are the closure and boundary of $[2, 5) \subset \mathbb{R}$? 14
(b) What is the closure and boundary of the closed ball of radius 2 around the origin in $\mathbb{R}^2$? 15

Continuity

Definition 9. Let $M_1$, $M_2$ be metric spaces, with $d_1$ and $d_2$ as their corresponding distance functions. A function $f : M_1 \to M_2$ is said to be continuous at $a \in M_1$ iff as $x \to a$, $f(x) \to f(a)$; this means: $\forall \epsilon > 0$, $\exists \delta > 0$ such that for every $x$ that satisfies $d_1(a, x) < \delta$ we have $d_2(f(a), f(x)) < \epsilon$; or, equivalently, $f(B_\delta(a)) \subset B_\epsilon(f(a))$. We say $f$ is continuous if it is continuous at every point in $M_1$.

Example 25. Prove that $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = 2x$ is continuous.

Proof: Fix an arbitrary point $p \in \mathbb{R}$. We will show $f$ is continuous at $p$, by showing that $\forall \epsilon > 0$, $\exists \delta > 0$ such that $\forall q \in B_\delta(p)$, $f(q) \in B_\epsilon(f(p))$.

Pick any $\epsilon > 0$. Let $\delta = \epsilon/2$. Then, for any $q \in B_\delta(p)$ we have: $d(f(p), f(q)) = |2p - 2q| = 2|p - q| < 2\delta = \epsilon$. So $f(q) \in B_\epsilon(f(p))$, as desired. Since $p$ was arbitrary, $f$ is continuous at every point in $\mathbb{R}$.

Example 26. Determine whether each of the following functions $f$ and $g$ from $\mathbb{R}$ to $\mathbb{R}$ is continuous at 0. (Support your answers informally, without rigorous proof.)

$$f(x) = \begin{cases} \sin(1/x) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases} \quad g(x) = \begin{cases} x \sin(1/x) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

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11 $(2, 5)$.
12 $(2, 5)$.
13 the open ball of radius 2 around the origin.
14 closure $= [2, 5]$; boundary $= \{2, 5\}$.
15 closure $= \text{itself}$; boundary $= \text{circle of radius 2 around the origin}$.