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Name: ____________________________________________
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Extra Credit problems do not carry any points; so do not spend any time on them unless you’re sure you’ve done your best with the problems that do carry points.

1. (a) (5 points) Give the definition of **homeomorphism**. Write complete and grammatically correct sentences.
   (b) (20 points) Prove \((2, 5) \simeq \mathbb{R}\). Just find an appropriate map; you do not need to prove it’s a homeomorphism. Give a brief explanation of how you come up with the map.

2. (a) (5 points) Give the definition of a **continuous** map between two metric spaces. Write complete and grammatically correct sentences.
   (b) (20 points) Suppose \((X_1, d_1)\) and \((X_2, d_2)\) are metric spaces, and suppose \(f: X_1 \to X_2\) is a function such that the preimage of every open set is open, i.e., for every open set \(A_2 \subseteq X_2\), \(f^{-1}(A_2)\) is open in \(X_1\). Prove that \(f\) is continuous according to the definition of continuity for metric spaces.

Extra Credit Problems

3. (No points) Recall the construction of the **Cantor Set**: Start with \([0, 1] \subset \mathbb{R}\). Remove its open middle third, i.e., \((1/3, 2/3)\). You’re left with \([0, 1/3] \cup [2/3, 1]\). Now remove the open middle third of each of the above two remaining closed intervals, i.e., remove \((1/9, 2/9)\) and \((7/9, 8/9)\). Keep repeating this process forever. What remains in the end is called the **Cantor Set**, which we denote as \(C\).
   
   (a) Is \(C\) open, closed, both, or neither, in \(\mathbb{R}\)?
   (b) Prove that every point in \(C\) is a limit point of \(C\). You may leave out tedious details and just give an outline of all the main ideas of the proof.

4. (No points) **Definition**  Let \((X, d)\) be a metric space. We say a sequence of points \(a_1, a_2, a_3, \cdots \in X\) converges to a point \(p \in X\) if \(\forall \epsilon > 0 \exists M\) such that \(\forall n > M, d(a_n, p) < \epsilon\). We write \(\lim_{n \to \infty} a_n = p\), and say the sequence \(a_1, a_2, a_3, \cdots\) is **convergent**.
   Let \((X_1, d_1)\) and \((X_2, d_2)\) be metric spaces. Prove that \(f: X_1 \to X_2\) is continuous iff for every convergent sequence \(a_1, a_2, a_3, \cdots \in X_1\), \(\lim_{n \to \infty} f(a_n) = f(\lim_{n \to \infty} a_n)\).