1. Let $f : \mathbb{R} \to \mathbb{R}$ be given by: $f(x) = \begin{cases} 1/2 & \text{if } x < 0 \\ 1/3 & \text{if } x \geq 0 \end{cases}$. Prove that $f$ is not continuous at 0.

2. In the following, just find a map each problem asks for, without proving continuity, injectivity, or surjectivity. Each of the following sets is assumed to come with the standard Euclidean metric.

(a) Let $M_1 \subset \mathbb{R}^2$ be the closed unit disk (i.e., $M_1 = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 \leq 1\}$). Let $M_2 \subset \mathbb{R}^2$ be the closed disk of radius 2 centered at the origin. Find a continuous bijection (one-to-one and onto map) $f : M_1 \to M_2$.

(b) Let $M_3 \subset \mathbb{R}^2$ be the closed disk of radius 1 centered at the point $(3, 4)$. Find a continuous bijection $f : M_1 \to M_3$.

(c) Let $M_4 \subset \mathbb{R}^2$ be the closed disk of radius 2 centered at the point $(3, 4)$. Find a continuous bijection $f : M_1 \to M_4$.

3. Suppose $M_1 = (X_1, d_1)$ and $M_2 = (X_2, d_2)$ are metric spaces. Let $b \in X_2$, and let $f : X_1 \to X_2$ be the constant map $f(x) = b$, $\forall x \in X_1$. Show that $f$ is continuous on $X_1$.

4. Suppose $M_1 = (X_1, d_1)$ and $M_2 = (X_2, d_2)$ are metric spaces, and suppose $f : X_1 \to X_2$ is a continuous function. Prove that $\forall a \in X_1$ and $\forall \epsilon > 0$, $\exists \delta > 0$ such that the ball of radius $\delta$ around $a$ is mapped under $f$ to inside the ball of radius $\epsilon$ around $f(a)$; i.e., $f(B_\delta(a)) \subseteq B_\epsilon(f(a))$.

5. Suppose $M_1 = (X_1, d_1)$ and $M_2 = (X_2, d_2)$ are metric spaces, and suppose $f : X_1 \to X_2$ is a continuous function. Prove that the preimage of any open set in $M_2$ is an open set in $M_1$; i.e., if $A_2 \subset X_2$ is open, then $A_1 = f^{-1}(A_2) \subset X_1$ is also open.

Extra Credit Problems

6. The Cantor Set.

Start with $[0, 1] \subset \mathbb{R}$. Remove its open middle third, i.e., $(1/3, 2/3)$. You’re left with $[0, 1/3] \cup [2/3, 1]$. Now remove the open middle third of each of the above two remaining closed intervals, i.e., remove $(1/9, 2/9)$ and $(7/9, 8/9)$.

What’s remaining now? What is the open middle third of each remaining piece?

Keep repeating this process forever. What remains in the end is called the Cantor Set, which we denote as $C$.

Is $C$ open, closed, both, or neither, in $\mathbb{R}$?

Prove that every point in $C$ is a limit point of $C$. 

1