1. Show by example that the union of an infinite collection of closed subsets of a metric space is not necessarily closed.

2. In a previous homework problem we showed that the union of any two closed subsets of a metric space is closed. Use this fact, and mathematical induction, to prove that the union of any finite collection of closed subsets of a metric space is closed.

3. In a previous homework problem we showed that the union of any collection of open subsets of a metric space is open. Use this fact to prove that the intersection of any collection of closed subsets of a metric space is closed.

Hint: Use de Morgan’s Law: Let \( \{A_{\alpha} \mid \alpha \in I\} \) be any collection of subsets of some fixed set, where \( I \) is an index set; then

\[
\left( \bigcap_{\alpha \in I} A_{\alpha} \right)^c = \bigcup_{\alpha \in I} A_{\alpha}^c.
\]

Summary of unions and intersections of open or closed subsets

- Union of any collection of open sets is open.
- Intersection of any finite number of open sets is open.
- Intersection of an infinite number of open sets may not be open.
- Intersection of any collection of closed sets is closed.
- Union of any finite number of closed sets is closed.
- Union of an infinite number of closed sets may not be closed.

4. Let \( A \subseteq \mathbb{R}^2 \) be given by: \( A = \{(x, y) \mid 1 < x^2 + y^2 \leq 2\} \cup \{(0, 0)\} \).
   
   (a) Draw a picture of \( A \).
   
   (b) Draw a picture of, and describe, using set notation, \( A^\circ \) (the interior of \( A \)).
   
   (c) Draw a picture of, and describe, using set notation, \( \overline{A} \) (the closure of \( A \)).
   
   (d) Draw a picture of, and describe, using set notation, \( \partial A \) (the boundary of \( A \)).
   
   (e) Find all limit points of \( A \) that are not in \( A \). Is \( (0, 0) \) a limit point of \( A \)? Why?

Extra Credit Problems

5. Let \( A \) be a subset of a metric space \( M \). Prove \( A^\circ \) is open.

6. Let \( A \) be a subset of a metric space \( M \). Prove \( A^\circ \) is the union of all subsets \( C \subseteq A \) which are open in \( M \). Prove \( \overline{A} \) is the intersection of all subsets \( C \) such that \( A \subseteq C \) and \( C \) is closed in \( M \).