Section 9: Classification of 2-manifolds.

Recall: How many connected 1-mfds are there?\(^1\) It turns out that there are a lot more connected 2-mfds; in fact, there are infinitely many of them. Nevertheless, we can classify them, which roughly means we can systematically list them all, without any repetitions. (We’ll soon have a better idea of what this means.) We will first concentrate only on 2-manifolds that are closed and can be embedded in \(\mathbb{R}^3\); next, those that are closed but cannot be embedded in \(\mathbb{R}^3\); and finally non-closed 2-mfds, but only compact ones. Non-compact 2-manifolds are more difficult to describe, and we’ll skip them.

Closed surfaces that are embeddable in \(\mathbb{R}^3\)

**Example 1.** What is the definition of a closed manifold?\(^2\) Which surfaces have we seen so far that are closed and can be embedded in \(\mathbb{R}^3\)? Think before reading the following theorem!

**Theorem 1.** Every closed 2-manifold that can be embedded in \(\mathbb{R}^3\) is homeomorphic to \(S^2\) or to an \(n\)-hole torus (= the connected sum of \(n\) tori) for some \(n \geq 1\). Proof: Omitted

**Definition 1.** Let \((X,d)\) be a metric space, and let \(A \subseteq X\). Given \(\epsilon > 0\), the \(\epsilon\)-neighborhood of \(A\) in \(X\) is defined as the set of all points in \(X\) whose distance is less than \(\epsilon\) from some point in \(A\):

\[
N_\epsilon(A) = \{ x \in X \mid d(x,a) < \epsilon \text{ for some } a \in A \}.
\]

**Example 2.** Let \(B\) be the horizontal line \(y = 2\) in \(\mathbb{R}^2\), and let \(\epsilon = 0.1\). What is \(N_\epsilon(B)\)? What is \(\partial(N_\epsilon(B))\), the boundary of the \(\epsilon\)-neighborhood of \(B\)?\(^3\)

**Example 3.** Let \(A = \{(x,y,z) \in \mathbb{R}^3 \mid 0 \leq x, y, z \leq 1\} \) and at least two of \(x,\,y,\,z\) are in the set \{0,1\}. Draw a picture of \(A\).\(^4\) Let \(F = \partial(N_{0.1}(A))\). Draw a picture of \(F\). \(F\) is a closed surface \(\subset \mathbb{R}^3\). So, according to the theorem above, \(F\) is homeomorphic to an \(n\)-hole torus for some \(n\) (since \(F\) is clearly not homeomorphic to \(S^2\)). You will find \(n\) in the homework assignment.

Closed surfaces that are not embeddable in \(\mathbb{R}^3\)

**Example 4.** Let \(X = [0,1]^2/\{(x,0) \sim (x,1), (0,y) \sim (1,y)\}\). Is \(X\) a 2-manifold? Is \(X\) embeddable in \(\mathbb{R}^3\)?\(^5\)

In the following questions, you should be able to answer whether or not the given quotient space is a manifold. Determining whether or not it can be embedded in \(\mathbb{R}^3\) is more difficult, and you may not be able to see why.

Let \(M = [0,1]^2/\{(0,y) \sim (1,1-y)\}\). Is \(M\) a 2-manifold? Can you embed \(M\) in \(\mathbb{R}^3\)?

Let \(K = [0,1]^2/\{(x,0) \sim (x,1), (0,y) \sim (1,1-y)\}\). Is \(K\) a 2-manifold? Can you embed \(K\) in \(\mathbb{R}^3\)?

Let \(P = [0,1]^2/\{(x,0) \sim (1-x,1), (0,y) \sim (1,1-y)\}\). Is \(P\) a 2-manifold? Can you embed \(P\) in \(\mathbb{R}^3\)?\(^6\)

**Definition 2.** \(M = [0,1]^2/\{(0,y) \sim (1,1-y)\}\) is called the Möbius band (or Möbius strip). \(K = [0,1]^2/\{(x,0) \sim (x,1), (0,y) \sim (1,1-y)\}\) is called the Klein bottle. \(P = [0,1]^2/\{(x,0) \sim (1-x,1), (0,y) \sim (1,1-y)\}\) is called the projective plane, more commonly denoted by \(\mathbb{R}P^2\).

**Remark.** The projective plane is often also referred to as the real projective plane. This is in contrast to the complex projective plane \(\mathbb{C}P^2\), which we will not be studying.

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\(^1\)Only four: \(S^1, [a,b], [a,b), (a,b).\)

\(^2\)Compact, with no boundary.

\(^3\)\(N_\epsilon(A) = \{ x \in \mathbb{R}^3 \mid 1.9 < y < 2.1 \}. \partial(N_\epsilon(A)) \) consists of the two lines \(y = 1.9\) and \(y = 2.1\).

\(^4\)\(A\) consists of the 12 edges of a cube.

\(^5\)Yes to both; \(X\) is a torus.

\(^6\)\(M,\,K,\,P\) are all 2-manifolds. Only \(M\) can be embedded in \(\mathbb{R}^3\).
Theorem 2. The Projective Plane and the Klein Bottle cannot be embedded in \( \mathbb{R}^3 \). Proof omitted.

Theorem 3. (1) The boundary of a Möbius band is a circle: \( \partial M \simeq S^1 \). (2) Gluing a Möbius band and a closed disk along their circle-boundaries yields a projective plane: \( M \cup \overline{D^2} \simeq \mathbb{RP}^2 \). (3) Gluing two Möbius bands along their circle-boundaries yields a Klein bottle: \( M \cup \partial M \simeq K \).

Proof. (Sketch)

(1) By definition, \( M = [0, 1]^2 / \{ (0, y) \sim (1, 1-y) \} \). Therefore, \( \partial M \) consists of those points in \( \partial([0, 1]^2) \) that are not identified with any other point—except that the four corners of the square are, after being pairwise identified, in \( \partial M \). So \( \partial M = ([0, 1] \times \{0\} \cup [0, 1] \times \{1\})/\{(0,0) \sim (1,1) , (1,0) \sim (0,1)\} \), which is homeomorphic to a circle.

(2) It’s enough to show \( M \simeq \mathbb{RP}^2 - D^2 \), as in the following diagrams.

(3) Homework.

Theorem 4. Every closed 2-manifold that cannot be embedded in \( \mathbb{R}^3 \) is homeomorphic to the connected sum of \( n \) projective planes for some \( n \geq 1 \). Proof: Omitted.

Definition 3. For \( n \geq 1 \), the \( n \)-hole torus is the connected sum of \( n \) tori, denoted by \( nT^2 \). Similarly, the connected sum of \( n \) projective planes is denoted by \( n\mathbb{RP}^2 \).

Corollary 5. Every closed 2-manifold is homeomorphic to either \( S^2 \) or \( nT^2 \) or \( n\mathbb{RP}^2 \), for some \( n \geq 1 \).

Example 5. According to the above corollary, \( T^2 \# \mathbb{RP}^2 \) is homeomorphic to either \( S^2 \) or \( nT^2 \) or \( n\mathbb{RP}^2 \), for some \( n \). Which is it? The next theorem answers this.

Theorem 6. \( T^2 \# \mathbb{RP}^2 \simeq 3 \mathbb{RP}^2 \). Proof: Homework.

Corollary 7. \( T^2 \# \mathbb{RP}^2 \simeq K \# \mathbb{RP}^2 \). Proof: Homework.

The above corollary may seem to suggest that \( T^2 \simeq K \), which is not true!

Theorem 8. A torus is not homeomorphic to a Klein bottle. Proof: Homework.

Theorem 9. \( S^2 \not\simeq \mathbb{RP}^2 \). Proof: Homework.

You will need the following definition for the homework assignment.

Definition 4. Let \( A \) be a subset of a connected topological space \( X \). To say \( A \) separates \( X \) means \( X - A \) is not connected.

Example 6. Does \( S^1 \) separate \( \mathbb{R}^2 \)? Does \( [0, \infty) \) separate \( \mathbb{R}^2 \)? Is there a non-separating embedded circle on the torus? How about a separating one? \(^7\)

\(^7\) Y, N, Y, Y. Draw pictures!