Section 7: Manifolds with boundary

Math 460 Topology

Review definitions of neighborhood, locally homeomorphic, and manifold.

Recall that, in the definition of manifold, we can replace “locally homeomorphic to an open ball in $\mathbb{R}^n$” with “locally homeomorphic to $\mathbb{R}^n$.”

Example 1. Is the open rectangle $(0,1) \times (0,2) \subset \mathbb{R}^2$ a manifold? Yes. Of what dimension? 2.

Is the closed rectangle $[0,1] \times [0,2] \subset \mathbb{R}^2$ a manifold? No. Why?

We’d like to say that the closed rectangle is a manifold with boundary. Before defining this, we need another definition.

Definition 1. The $n$-dimensional upper half-space is defined as

$$\mathbb{R}_+^n = \{(x_1, \ldots, x_n) \in \mathbb{R}^n | x_n \geq 0\}$$

When $n = 2$, $\mathbb{R}_+^2$ is also called the upper half-plane.

Example 2. Draw a picture of what each of $\mathbb{R}_+^1$ and $\mathbb{R}_+^2$ looks like.

Example 3. Let $X = B_1(0,0) \cap \mathbb{R}_+^2$. Is $X$ homeomorphic to $\mathbb{R}_+^2$? Does every point in $X$ have a neighborhood that’s homeomorphic to either $\mathbb{R}^2$ or $\mathbb{R}_+^2$?

Definition 2. (Overrides previous definition of manifold) A topological space $X$ is called an $n$-dimensional manifold (n-mfd for short) if it is Hausdorff, Second Countable, and every point $x \in X$ has a neighborhood that is homeomorphic to $\mathbb{R}^n$ or $\mathbb{R}_+^n$. A point that has a neighborhood homeomorphic to $\mathbb{R}_+^n$ but not to $\mathbb{R}^n$ is called a boundary point. The set of all such points (if any) is called the boundary of $X$, denoted by $\partial X$. If $\partial X \neq \phi$, then, for emphasis, $X$ is sometimes called a manifold with boundary.

Remark. Depending on the context, the term boundary can have two different meanings: when applied to a subset $A$ of a topological space, it means $\overline{A} - A^o$; but when applied to a manifold, it is defined according to the above definition. For a given topological space that’s also a mfd, these two different types of boundary may happen to be the same set of points, but most often they are not! (Thus, the symbol $\partial$ has at least three different meanings in mathematics: two types of boundary, plus partial derivative.)

Example 4. Let $X = ([0,1] \times [0,1])/\{(0,y) \sim (1,y)\}$. Draw a picture of $X$. Is $X$ a manifold? Yes. Is it a manifold with boundary? Yes. What is $\partial X$?

Example 5. Let $X = [0,1] \times [0,1]/\{(0,y) \sim (\frac{1}{2},y)\}$. Draw a picture of $X$. Is $X$ a manifold?

Theorem 1. (Classification of 1-manifolds) Every 1-manifold is homeomorphic to $[0,1]$ or $(0,1)$ or $[0,1)$ or $S^1$.

Idea of Proof: What things can you create by joining or overlapping line segments end-to-end? Only these four.

We are often interested in studying manifolds that are compact and have no boundary. (Why? One reason is that non-compact manifolds are usually more difficult to study, or at least different very from compact ones.) There is a name for such manifolds:

1Yes to both; why?
2$X$ is homeomorphic to a cylinder, and it’s boundary is homeomorphic two disjoint circles: $([0,1] \times \{0\}/\{(0,0) \sim (1,0)\}) \cup ([0,1] \times \{1\}/\{(0,1) \sim (1,1)\})$.
3No, why?
**Definition 3.** A manifold is said to be **closed** if it is compact and has no boundary.

**Remark.** Do not confuse the two (very) different meanings of closed; they depend on the context: A subset $A$ of a topological space $X$ is closed if $X - A$ is open in $X$ (i.e., $X - A$ in $T$). A manifold is closed if it’s compact and has no boundary.

**Example 6.** Which of the four 1-mfds are closed? Why is each of the others not closed?  

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**Hausdorff Spaces**

Definition: A topological space $X$ is said to be **Hausdorff** iff every pair of distinct points $x_1, x_2 \in X$ can be separated by open sets, i.e., there exist disjoint open sets $U_1, U_2 \subseteq X$ such that $x_i \in U_i$.

**Example 7.** Determine whether each of the following is Hausdorff.

(a) $\mathbb{R}^2$ with the standard topology (induced by the Euclidean metric).

(b) $\mathbb{R}^2$ with the discrete topology.

(c) $\mathbb{R}^2$ with the indiscrete topology.  

**Example 8.** Which of the following are manifolds? Why?

(a) $([0, 2] \cup [5, 7])/\{\forall x \in [0, 1], x \sim (x + 5)\}$.  

(b) $([0, 2] \cup [5, 7])/\{\forall x \in [0, 1], x \sim (x + 5)\}$.

Ans: Every point does have a neighborhood that’s homeomorphic to $\mathbb{R}$; nevertheless, this is not a mfd! Why?

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4 Only $S^1$ is closed. $[0, 1]$ has boundary. $(0, 1)$ isn’t compact. $[0, 1)$ has boundary and isn’t compact.  

5 Yes, yes, no. Why?  

6 Not a mfd, since the point $[1] = [6] = \{1, 6\}$ in the quotient space does not have a neighborhood that’s homeomorphic to $\mathbb{R}^n$ or $\mathbb{R}_n^+$ for any $n$.  

7 Because it’s not Hausdorff: the points 1 and 6 cannot be separated by open sets.