**Section 1:** Metric spaces; open and closed sets; limit points; interior, closure, boundary; continuity.

**Math 460 Topology**

**Definition 1.** A **metric space** $M$ is a set $X$ and a function $d : X \times X \to [0, \infty)$ such that $\forall x, y, z \in X$

1. $d(x, y) = 0$ iff $x = y$;
2. $d(x, y) = d(y, x)$ (d is symmetric);
3. $d(x, z) \leq d(x, y) + d(y, z)$ (triangle inequality).

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**Example 1.** $\mathbb{R}$ with the **Euclidean metric** (the “standard” metric):

$X = \mathbb{R}$, $d(x, y) = |x - y|$. Why is this a metric space? What if we replace $d$ with $d(x, y) = x - y$; would we have a metric space?

**Example 2.** $\mathbb{R}$ with the **discrete metric**, denoted $\mathbb{R}_d$:

$X = \mathbb{R}$, $d(x, y) = \begin{cases} 0 & \text{if } x = y \\ 1 & \text{if } x \neq y \end{cases}$. Why is this a metric space? How about $d(x, y) = 0 \forall x, y$?

**Example 3.** $\mathbb{R}^n$ with the **Euclidean metric**:

$X = \mathbb{R} \times \cdots \times \mathbb{R}$ ($n$ times), for $x = (x_1, \ldots, x_n)$, $y = (y_1, \ldots, y_n)$, $d(x, y) = \sqrt{(x_1 - y_1)^2 + \cdots + (x_n - y_n)^2}$. Why is this a metric space? (Do in HW)

**Example 4.** $\mathbb{R}^2$ with the **taxicab metric**:

$X = \mathbb{R}^2$, for $a = (a_1, a_2)$, $b = (b_1, b_2)$, $d(a, b) = |a_1 - b_1| + |a_2 - b_2|$. Why is this a metric space?

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**Note.**

1. Unless stated otherwise, whenever we refer to $\mathbb{R}$ as a metric space without stating what the metric function $d$ is, we mean “$\mathbb{R}$ with the Euclidean metric.”

2. For a metric space $M = (X, d)$, $X$ is called the **underlying set**. We will often abuse notation and write $M$ instead of $X$, or vice versa; for example, we may write $x \in M$ instead of $x \in X$; or we may refer to $X$ as a metric space, when it’s really $M = (X, d)$ that’s a metric space.

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**Definition 2.** Given a metric space $M$, a point $x \in M$, and a real number $r \geq 0$, the **ball** of radius $r$ around $x$ is defined as

$$B_r(x) = \{ y \in M | d(x, y) < r \}$$

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**Example 5.** In $\mathbb{R}$ with the Euclidean metric, $B_2(1) = ?$ \(^1\)

**Example 6.** In $\mathbb{R}^2$ with the Euclidean metric, what does $B_2(1, 2)$ look like? (Strictly speaking, we should write $B_2((1, 2))$; but too many parentheses can make in difficult to read, so we slightly abuse notation and write only one set of parentheses.) How about $B_2(1, 2) \subset \mathbb{R}^3$, what does it look like?

**Example 7.** In $\mathbb{R}_d$, what is $B_3(8)$? What is $B_{0.5}(8)$? \(^2\)

**Example 8.** In $\mathbb{R}^2$ with the taxicab metric, what does $B_4(0, 0)$ look like?

**Example 9.** Is there a metric on $\mathbb{R}^2$ for which $B_1(0, 0) = (-1, 1) \times (-1, 1)$? \(^3\)

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**Definition 3.** A subset $A$ of a metric space $M$ is said to be **open** in $M$ iff $\forall x \in A$, $\exists r > 0$ such that $B_r(x) \subset A$.

\(^1\)The open interval from $-1$ to 3: $(-1, 3)$.

\(^2\) $B_3(8) = \mathbb{R}$; $B_{0.5}(8) = \{8\}$.

\(^3\) $d(a, b) = \max\{|a_1 - b_1|, |a_2 - b_2|\}$. 

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Definition 4. Let $A$ be a subset of a metric space $M$. The complement of $A$ is $A^c = M - A$. $A$ is said to be closed in $M$ iff its complement $A^c$ is open in $M$.

Example 13. $(-\infty, -1] \cup [1, \infty)$ is closed in $\mathbb{R}$. Why?

Example 14. Is $(-\infty, -1]$ closed in $\mathbb{R}$?  

Example 15. Is $[-1, 1]$ closed in $\mathbb{R}$? 

Example 16. $[-1, 1)$ is neither open nor closed in $\mathbb{R}$. Why?

Example 17. $\mathbb{R}$ is open in $\mathbb{R}$. $\phi$ is open in $\mathbb{R}$. Why?

Example 18. $\mathbb{R}$ is closed in $\mathbb{R}$. $\phi$ is closed in $\mathbb{R}$. Why?

Example 19. Is $\mathbb{R}$ open or closed or neither in $\mathbb{R}^2$? 

Example 20. Find an open set in $\mathbb{R}_d$. Find a closed set in $\mathbb{R}_d$. 

(Quote from Munkres’s book, Topology: Q: “What’s the difference between a door and a set?” A: “A door is always either open or closed.”)

For emphasis, $B_r(x)$ is sometimes called the open ball of radius $r$ around $x$. In contrast, we have:

Definition 5. The closed ball of radius $r$ around $x$ is defined as

$$B_r(x) = \{ y \in M \mid d(x, y) \leq r \}$$

Example 21. Draw the open and closed balls of radius $5$ around the point $2$ in $\mathbb{R}$. Draw the open and closed balls of radius $5$ around the point $(2, 5)$ in $\mathbb{R}^2$.

Definition 6. Let $A$ be a subset of a metric space $M$. A point $x \in M$ is said to be a limit point of $A$ iff every ball around $x$ contains a point of $A$ other than $x$.

(Synonyms of limit point: cluster point; accumulation point.)

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4Because there is no positive $r$ for which $B_r(1) \subset (-1, 1]$.

5Yes. Why?

6Yes. Why?

7Closed. Why?

8Each of $\mathbb{R}_d$ and $\phi$ is both open and closed.
Example 22. Let $M = \mathbb{R}$, $A = [0, 2)$. Which of the points $x = 0, 1, 2, 3$ are limit points of $A$? Why? \(^9\) What if $A = [0, 1] \cup \{2\}$? \(^10\)

(Equivalent definition of limit point: $x$ is a limit point of $A$ iff $\forall \epsilon > 0$, $\exists y \in A - \{x\}$ such that $d(x, y) < \epsilon$.)

Theorem 1. A subset $A$ of a metric space $M$ is closed iff it contains all its limit points.

Proof. “$\Rightarrow$” : Suppose $A$ is closed. Then, by definition, $A^c$ is open. Let $x$ be a limit point of $A$. We want to show $x \in A$. By definition of limit point, every open ball around $x$ intersects $A - \{x\}$; therefore no open ball around $x$ is entirely contained in $A^c$. This implies $x \notin A^c$, since if $x$ were in $A^c$, then there would be an open ball around $x$ contained entirely in $A^c$ (since $A^c$ is open). Finally, since $x \notin A^c$, $x$ must be in $A$, as desired.

“$\Leftarrow$” : (Do yourself!) \(\square\)

Definition 7. Given a subset $A$ of a metric space $M$, its interior $A^o$ is defined as the set of all points $x \in A$ such that some open ball around $x$ is a subset of $A$.

Example 23. (a) What is the interior of $[2, 5) \subset \mathbb{R}$? \(^11\)
(b) What is the interior of $(2, 5) \subset \mathbb{R}$? \(^12\)
(c) What is the interior of the closed ball of radius 2 around the origin in $\mathbb{R}^2$? \(^13\)

Definition 8. Given a subset $A$ of a metric space $M$, its closure $\overline{A}$ is defined as $A$ union the set of all limit points of $A$. The boundary of $A$ is defined as $\partial A = \overline{A} - A^o$.

Example 24. (a) What are the closure and boundary of $[2, 5) \subset \mathbb{R}$? \(^14\)
(b) What is the closure and boundary of the closed ball of radius 2 around the origin in $\mathbb{R}^2$? \(^15\)

Continuity

Definition 9. Let $M_1$, $M_2$ be metric spaces, with $d_1$ and $d_2$ as their corresponding distance functions. A function $f : M_1 \to M_2$ is said to be continuous at $a \in M_1$ iff as $x \to a$, $f(x) \to f(a)$; this means: $\forall \epsilon > 0$, $\exists \delta > 0$ such that for every $x$ that satisfies $d_1(a, x) < \delta$ we have $d_2(f(a), f(x)) < \epsilon$; or, equivalently, $f(B_\delta(a)) \subset B_\epsilon(f(a))$. We say $f$ is continuous if it is continuous at every point in $M_1$.

Example 25. Prove that $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = 2x$ is continuous.

Proof: Fix an arbitrary point $p \in \mathbb{R}$. We will show $f$ is continuous at $p$, by showing that $\forall \epsilon > 0$, $\exists \delta > 0$ such that $\forall q \in B_\delta(p)$, $f(q) \in B_\epsilon(f(p))$.\(^9\) 0, 1 and 2. \(^10\) 0 and 1. \(^11\) (2, 5). \(^12\) (2, 5). \(^13\) the open ball of radius 2 around the origin. \(^14\) closure = $[2, 5]$; boundary = $\{2, 5\}$. \(^15\) closure = itself; boundary = circle of radius 2 around the origin.
Pick any $\epsilon > 0$. Let $\delta = \epsilon / 2$. Then, for any $q \in B_\delta(p)$ we have: $d(f(p), f(q)) = |2p - 2q| = 2|p - q| < 2\delta = \epsilon$. So $f(q) \in B_\epsilon(f(p))$, as desired. Since $p$ was arbitrary, $f$ is continuous at every point in $\mathbb{R}$.

**Example 26.** Determine whether each of the following functions $f$ and $g$ from $\mathbb{R}$ to $\mathbb{R}$ is continuous at $0$. (Support your answers informally, without rigorous proof.)

\[ f(x) = \begin{cases} \sin(1/x) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases} \quad g(x) = \begin{cases} x \sin(1/x) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases} \]