Review definitions of neighborhood, locally homeomorphic, and manifold.

Recall that, in the definition of manifold, we can replace “locally homeomorphic to an open ball in $\mathbb{R}^n$” with “locally homeomorphic to $\mathbb{R}^n$."

Example 1. Is the open rectangle $(0, 1) \times (0, 2) \subset \mathbb{R}^2$ a manifold? Yes. Of what dimension? 2.

Is the closed rectangle $[0, 1] \times [0, 2] \subset \mathbb{R}^2$ a manifold? No. Why?

We’d like to say that the closed rectangle is a manifold with boundary. Before defining this, we need another definition.

Definition 1. The $n$-dimensional upper half-space is defined as

$$\mathbb{R}^n_+ = \{(x_1, \ldots, x_n) \in \mathbb{R}^n \mid x_n \geq 0\}$$

When $n = 2$, $\mathbb{R}^2_+$ is also called the upper half-plane.

Example 2. Draw a picture of what each of $\mathbb{R}^1_+$ and $\mathbb{R}^2_+$ looks like.

Example 3. Let $X = B_1(0, 0) \cap \mathbb{R}^2_+$. Is $X$ homeomorphic to $\mathbb{R}^2_+$? Does every point in $X$ have a neighborhood that’s homeomorphic to either $\mathbb{R}^2$ or $\mathbb{R}^2_+$?

Definition 2. (Overides previous definition of manifold) A topological space $X$ is called an $n$-dimensional manifold ($n$-mfd for short) if it is Hausdorff, Second Countable, and every point $x \in X$ has a neighborhood that is homeomorphic to $\mathbb{R}^n$ or $\mathbb{R}^n_+$. A point that has a neighborhood homeomorphic to $\mathbb{R}^n_+$ but has no neighborhood that’s homeomorphic to $\mathbb{R}^n$ is called a boundary point. The set of all such points (if any) is called the boundary of $X$, denoted by $\partial X$. If $\partial X \neq \emptyset$, then, for emphasis, $X$ is sometimes called a manifold with boundary.

Remark. Depending on the context, the term boundary can have two different meanings: when applied to a subset $A$ of a topological space, it means $\overline{A} - A^c$; but when applied to a manifold, it is defined according to the above definition. For a given topological space that’s also a mfd, these two different types of boundary may happen to be the same set of points, but most often they are not! (Thus, the symbol $\partial$ has at least three different meanings in mathematics: two types of boundary, plus partial derivative.)

Example 4. Let $X = ([0, 1] \times [0, 1]) / \{(0, y) \sim (1, y)\}$. Draw a picture of $X$. Is $X$ a manifold? Yes. Is it a manifold with boundary? Yes. What is $\partial X$? 2

Example 5. Let $X = [0, 1] \times [0, 1] / \{(0, y) \sim (\frac{1}{2}, y)\}$. Draw a picture of $X$. Is $X$ a manifold? 3

Theorem 1. (Classification of 1-manifolds) Every connected 1-manifold is homeomorphic to $[0, 1]$ or $(0, 1)$ or $S^1$.

Idea of Proof: What things can you create by joining or overlapping line segments end-to-end? Only these four.

We are often interested in studying manifolds that are compact and have no boundary. (Why? One reason is that non-compact manifolds are usually more difficult to study, or at least different very from compact ones.) There is a name for such manifolds:

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1Yes to both; why?

2$X$ is homeomorphic to a cylinder, and it’s boundary is homeomorphic to two disjoint circles: $([0, 1] \times \{0\} / \{(0, 0) \sim (1, 0)\}) \cup ([0, 1] \times \{1\} / \{(0, 1) \sim (1, 1)\})$.

3No, why?
Definition 3. A manifold is said to be **closed** if it is compact and has no boundary.

**Remark.** Do not confuse the two (very) different meanings of *closed*; they depend on the context: A subset $A$ of a topological space $X$ is closed if $X - A$ is open in $X$ (i.e., $X - A$ is in $\mathcal{T}$). A manifold is closed if it’s compact and has no boundary.

**Example 6.** Which of the four connected 1-mfds are closed? Why is each of the others not closed? 4

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**Hausdorff Spaces**

Definition: A topological space $X$ is said to be **Hausdorff** iff every pair of distinct points $x_1, x_2 \in X$ can be **separated** by open sets, i.e., there exist disjoint open sets $U_1, U_2 \subseteq X$ such that $x_i \in U_i$.

**Example 7.** Determine whether each of the following is Hausdorff.

- (a) $\mathbb{R}^2$ with the standard topology (induced by the Euclidean metric).
- (b) $\mathbb{R}^2$ with the discrete topology.
- (c) $\mathbb{R}^2$ with the indiscrete topology. 5

**Example 8.** Which of the following are manifolds? Why?

- (a) $([0, 2] \cup [5, 7])/\{\forall x \in [0, 1], x \sim (x + 5)\}$. 6
- (b) $([0, 2] \cup [5, 7])/\{\forall x \in [0, 1), x \sim (x + 5)\}$.

Answer to (b): Every point does have a neighborhood that’s homeomorphic to $\mathbb{R}$; nevertheless, this is not a manifold! Why? 7

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4 Only $S^1$ is closed. $[0, 1]$ has boundary. $[0, 1)$ isn’t compact. $[0, 1)$ has boundary and isn’t compact.

5 Yes, yes, no. Why?

6 Not a mfd, since the point $[1] = [6] = \{1, 6\}$ in the quotient space does not have a neighborhood that’s homeomorphic to $\mathbb{R}^n$ or $\mathbb{R}^n$ for any $n$.

7 Because it’s not Hausdorff: the points 1 and 6 cannot be separated by open sets.