**Section 1:** Metric spaces; open and closed sets; limit points; interior, closure, boundary; continuity.

**Definition 1.** A **metric space** \( M \) consists of a set \( X \) and a **distance function** \( d : X \times X \to [0, \infty) \) such that \( \forall x, y, z \in X \)
1. \( d(x, y) = 0 \) iff \( x = y \);
2. \( d(x, y) = d(y, x) \) (\( d \) is symmetric);
3. \( d(x, z) \leq d(x, y) + d(y, z) \) (triangle inequality).

**Example 1.** \( \mathbb{R} \) with the **Euclidean metric** (the “standard” metric):
\( X = \mathbb{R} \), \( d(x, y) = |x - y| \). Why is this a metric space?
If instead we had \( d(x, y) = x - y \), would we still have a metric space?

**Example 2.** \( \mathbb{R} \) with the **discrete metric**, denoted \( \mathbb{R}_d \):
\( X = \mathbb{R} \), \( d(x, y) = \begin{cases} 0 & \text{if } x = y \\ 1 & \text{if } x \neq y \end{cases} \). Why is this a metric space?
What if we let \( d(x, y) = 0 \) for all \( x, y \), is it still a metric space?

**Example 3.** \( \mathbb{R}^n \) with the **Euclidean metric** :
\( X = \mathbb{R} \times \cdots \times \mathbb{R} \) \( (n \text{ times}); for \( x = (x_1, \ldots, x_n) \), \( y = (y_1, \ldots, y_n) \), \( d(x, y) = \sqrt{(x_1 - y_1)^2 + \cdots + (x_n - y_n)^2} \).
Why is this a metric space? Conditions 1 and 2 of the definition (above) are clearly satisfied. Condition 3 is the well-known triangle inequality (skip proof).

**Example 4.** \( \mathbb{R}^2 \), for \( a = (a_1, a_2) \), \( b = (b_1, b_2) \), \( d(a, b) = |a_1 - b_1| + |a_2 - b_2| \). Why is this a metric space? (HW)

**Note.**

1. Unless stated otherwise, whenever we refer to \( \mathbb{R} \) as a metric space without stating what the distance function \( d \) is, we mean “\( \mathbb{R} \) with the Euclidean metric.”

2. For a metric space \( M = (X, d) \), \( X \) is called the **underlying set**. We will often abuse notation and write \( M \) instead of \( X \), or vice versa; for example, we may write \( x \in M \) instead of \( x \in X \); or we may refer to \( X \) as a metric space, when it’s really \( M = (X, d) \) that’s a metric space.

**Definition 2.** Given a metric space \( M \), a point \( x \in M \), and a real number \( r > 0 \), the **ball** of radius \( r \) around \( x \) is defined as
\[
B_r(x) = \{ y \in M \mid d(x, y) < r \}
\]

**Example 5.** In \( \mathbb{R} \) with the Euclidean metric, \( B_2(1) = ? \)

**Example 6.** In \( \mathbb{R}^2 \) with the Euclidean metric, what does \( B_2(1, 2) \) look like? (Strictly speaking, we should write \( B_2((1, 2)) \); but too many parentheses can make in difficult to read, so we slightly abuse notation and write only one set of parentheses.) How about \( B_2(1, 2) \subset \mathbb{R}^3 \), what does it look like?

**Example 7.** In \( \mathbb{R}_d \), what is \( B_3(8) \)? What is \( B_{0.5}(8) \)?

**Example 8.** In \( \mathbb{R}^2 \) with the taxicab metric, what does \( B_1(0, 0) \) look like?

**Example 9.** Is there a metric on \( \mathbb{R}^2 \) for which \( B_1(0, 0) = (-1, 1) \times (-1, 1) \)?

**Definition 3.** A subset \( A \) of a metric space \( M \) is said to be **open** in \( M \) iff \( \forall x \in A \), \( \exists r > 0 \) such that \( B_r(x) \subset A \).

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1. The open interval from \(-1\) to \(3\): \((-1, 3)\).
2. \( B_2(8) = \mathbb{R}; B_{0.5}(8) = \{8\} \).
3. \( d(a, b) = \max\{|a_1 - b_1|, |a_2 - b_2|\} \).
Example 10. The interval \((-1, 1]\) is not open in \(\mathbb{R}\). Why?  

Example 11. The interval \((-1, 1)\) is an open subset of \(\mathbb{R}\). Why? 

Proof: Given an arbitrary \(x \in (-1, 1)\), let \(r = \min\{d(x, 1), d(x, -1)\}\). Then, we prove as follows that \(B_r(x) \subset (-1, 1)\). Let \(y \in B_r(x)\); we'll show \(y \in (-1, 1)\). We will do so by showing that \(d(0, y) < 1\). By definition of \(B_r(x)\), \(d(x, y) < r\); so \(d(x, y) < \min\{d(x, 1), d(x, -1)\}\); so \(d(x, y) < d(x, 1)\) and \(d(x, y) < d(x, -1)\). By the triangle inequality, \(d(0, y) \leq d(0, x) + d(x, y)\). So, \(d(0, y) < d(0, x) + d(x, 1)\) and \(d(0, y) < d(0, x) + d(x, -1)\). If \(x \geq 0\), then the right hand side of the first inequality equals 1. So either way, \(d(0, y) < 1\), as desired. We showed that for every \(x \in (-1, 1)\), there is a positive \(r\) such that \(B_r(x) \subset (-1, 1)\). So by the definition of open, \((-1, 1)\) is an open subset of \(\mathbb{R}\).

Example 12. Is the interval \((2, \infty)\) open in \(\mathbb{R}\)? Yes. Why?

Definition 4. Let \(A\) be a subset of a metric space \(M\). The **complement** of \(A\) is \(A^c = M - A\). \(A\) is said to be **closed** in \(M\) iff its complement \(A^c\) is open in \(M\).

Example 13. \((-\infty, -1] \cup [1, \infty)\) is closed in \(\mathbb{R}\). Why?

Example 14. Is \((-\infty, -1]\) closed in \(\mathbb{R}\)?  

Example 15. Is \([-1, 1]\) closed in \(\mathbb{R}\)?  

Example 16. \([-1, 1)\) is neither open nor closed in \(\mathbb{R}\). Why?

Example 17. \(\mathbb{R}\) is open in \(\mathbb{R}\). Why? \(\phi\) is open in \(\mathbb{R}\). Why?

Example 18. \(\mathbb{R}\) is closed in \(\mathbb{R}\). \(\phi\) is closed in \(\mathbb{R}\). Why?

Example 19. Is the \(x\)-axis open or closed or neither in \(\mathbb{R}^2\)?  

Example 20. Find an open set in \(\mathbb{R}_d\). Find a closed set in \(\mathbb{R}_d\).  

(Quote from Munkres's book, Topology: Q: “What’s the difference between a door and a set?” A: “A door is always either open or closed.”)

For emphasis, \(B_r(x)\) is sometimes called the open ball of radius \(r\) around \(x\). In contrast, we have:

**Definition 5.** The **closed ball** of radius \(r\) around \(x\) is defined as 

\[
B_r(x) = \{y \in M \mid d(x, y) \leq r\}
\]

Example 21. Draw the open and closed balls of radius 5 around the point 2 in \(\mathbb{R}\). Draw the open and closed balls of radius 5 around the point \((2, 5)\) in \(\mathbb{R}^2\).

**Definition 6.** Let \(A\) be a subset of a metric space \(M\). A point \(x \in M\) is said to be a **limit point** of \(A\) iff every ball around \(x\) contains a point of \(A\) other than \(x\).

(Synonyms of limit point: cluster point; accumulation point.)

**Example 22.** Let \(M = \mathbb{R}, A = [0, 2]\). Which of the points \(x = 0, 1, 2, 3\) are limit points of \(A\)? Why?  

What if \(A = [0, 1] \cup \{2\}\)?  

(Equivalent definition of limit point: \(x\) is a limit point of \(A\) iff \(\forall \epsilon > 0, \exists y \in A - \{x\} \) such that \(d(x, y) < \epsilon\).)

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4Because there is no positive \(r\) for which \(B_r(1) \subset (-1, 1)\).

5Yes. Why?

6Yes. Why?

7Closed. Why?

8Each of \(\mathbb{R}_d\) and \(\phi\) is both open and closed.

9\(0, 1\) and \(2\).

10\(0\) and \(1\).
Theorem 1. A subset \( A \) of a metric space \( M \) is closed iff it contains all its limit points.

Proof. \( \implies \) Suppose \( A \) is closed. Then, by definition, \( A^c \) is open. Let \( x \) be a limit point of \( A \). We want to show \( x \in A \). By definition of limit point, every open ball around \( x \) intersects \( A - \{x\} \); therefore no open ball around \( x \) is entirely contained in \( A^c \). This implies \( x \not\in A^c \), since if \( x \) were in \( A^c \), then there would be an open ball around \( x \) contained entirely in \( A^c \) (since \( A^c \) is open). Finally, since \( x \not\in A^c \), \( x \) must be in \( A \), as desired.

\( \impliedby \) (Do yourself!)

Definition 7. Given a subset \( A \) of a metric space \( M \), its interior \( A^\circ \) is defined as the set of all points \( x \in A \) such that some open ball around \( x \) is a subset of \( A \). (\( A^\circ \) is also written as \( \text{Int } A \) or \( \text{int}(A) \).)

Example 23. (a) What is the interior of \([2,5) \subset \mathbb{R} \)?

(b) What is the interior of \((2,5) \subset \mathbb{R} \)?

(c) What is the interior of the closed ball of radius 2 around the origin in \( \mathbb{R}^2 \)?

Definition 8. Given a subset \( A \) of a metric space \( M \), its closure \( \overline{A} \) is defined as \( A \) union the set of all limit points of \( A \). The boundary of \( A \) is defined as \( \partial A = \overline{A} - A^\circ \).

Example 24. (a) What are the closure and boundary of \([2,5) \subset \mathbb{R} \)?

(b) What is the closure and boundary of the closed ball of radius 2 around the origin in \( \mathbb{R}^2 \)?

Continuity

Definition 9. Let \( M_1, M_2 \) be metric spaces, with \( d_1 \) and \( d_2 \) as their corresponding distance functions. A function \( f : M_1 \to M_2 \) is said to be continuous at \( a \in M_1 \) iff as \( x \to a \), \( f(x) \to f(a) \); this means: \( \forall \epsilon > 0, \exists \delta > 0 \) such that for every \( x \) that satisfies \( d_1(a, x) < \delta \) we have \( d_2(f(a), f(x)) < \epsilon \). We say \( f \) is continuous if \( f \) is continuous at every point in \( M_1 \).

Example 25. Prove that \( f : \mathbb{R} \to \mathbb{R} \) defined by \( f(x) = 2x \) is continuous.

Proof: Fix an arbitrary point \( a \in \mathbb{R} \). We will show \( f \) is continuous at \( a \) by showing that \( \forall \epsilon > 0, \exists \delta > 0 \) such that for every \( x \) that satisfies \( d(a, x) < \delta \) we have \( d(f(a), f(x)) < \epsilon \).

Pick any \( \epsilon > 0 \). Let \( \delta = \epsilon/2 \). Then, for every \( x \) that satisfies \( d(a, x) < \delta \) we have: \( |a - x| < \delta \), so \( |2a - 2x| < 2\delta \), so \( d(f(a), f(x)) < \epsilon \), as desired. Since \( a \) was an arbitrary point, \( f \) is continuous at every point in \( \mathbb{R} \).

Example 26. Determine whether each of the following functions \( f \) and \( g \) from \( \mathbb{R} \) to \( \mathbb{R} \) is continuous at 0. (Support your answers informally, without rigorous proof.)

\[
\begin{align*}
  f(x) &= \begin{cases} 
    \sin(1/x) & \text{if } x \neq 0 \\
    0 & \text{if } x = 0
  \end{cases} \\
  g(x) &= \begin{cases} 
    x\sin(1/x) & \text{if } x \neq 0 \\
    0 & \text{if } x = 0
  \end{cases}
\end{align*}
\]

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\(^{11}(2,5)\).

\(^{12}(2,5)\).

\(^{13}\)the open ball of radius 2 around the origin.

\(^{14}\)closure = \([2,5]\); boundary = \((2,5)\).

\(^{15}\)closure = itself; boundary = circle of radius 2 around the origin.