HW 26 #1:
(a) Show \((0, 1) \cong (0, \infty)\).
\(\text{Pf} \) Define \(f : (0, 1) \to (0, \infty)\) 
by \(f(x) = \frac{1}{x} - 1\). Then 
f is a bij (PF omitted) 
& \(f\) takes rationals to rationals.

(b) Let \(T = \{\langle x, y, z \rangle \mid x, y, z \in \mathbb{N}\}\). 
Show \(T \cong \mathbb{N}\).
\(\text{Pf} \) Let \(f : \mathbb{N} \times \mathbb{N} \to \mathbb{N}\) be given by 
f(m, n) = \(2^m (2n+1) - 1\).
By previous HW, \(f\) is a bijection.
Define \(g : T \to \mathbb{N}\) by 
g(\langle x, y, z \rangle) = f(f(x, y), z).
Then \(g\) is a bij (PF omitted).
So \(T \cong \mathbb{N}\).

(c) Show \{finite strings of English alphabet\} 
is equinumerous to \(\mathbb{N}\).
\(\text{Pf} \) List the strings by increasing 
length, and within each group of 
strings of same length, order 
them lexicographically ("dictionary 
order"). Then assign the
Lemma: let \( x, y, z \) be sets s.t. \( x \notin z, y \in z \). Then 
\[ z \cong (z - y) \cup \{x\}. \]

Proof: Define \( f: z \to (z - y) \cup \{x\} \) by:
\[ f(a) = \begin{cases} 
  a & \text{if } a \neq y \\
  x & \text{if } a = y 
\end{cases} \]

Then \( f \) is a bijection (Proof: HW). \( \square \)

---

\[ \text{P.138 #6} \quad \text{let } K \text{ be a nonzero cardinal, show } \not\exists A \text{ a set containing all sets of cardinality } K. \]

Proof: Suppose toward contradiction there is such a set, \( A \). It's enough to show \( \forall x (x \in UA) \), for then \( UA \) would be a set that contains every set, which we've already shown is impossible.

Let \( x \) be any set. With \( x \in UA \).
1. \( \text{card } K = K \), by def of \( \text{card} \).
2. \( K \in A \), by def of \( A \).

If \( x \in K \), then \( x \in UA \) by 2 & def of \( U \), so we're done. So assume \( x \notin K \).
3. \( K \neq 0 \), by hypothesis.
4. \( \exists y (y \in K) \), by 3.
5. Let \( \alpha = (K - y) \cup \{x\} \).
6. \( \text{card } \alpha = \text{card } K = K \) by the lemma below.
7. \( \alpha \in A \), by 6 & def of \( A \).
8. \( x \in \alpha \), by 5 & def of \( U \).
9. \( x \in UA \), by 7 & 8.

So we've shown \( UA \) contains all sets, \( \not\exists A \). \( \square \)
P. 138 #7  Spse A is finite, f: A → A. Show f is 1-1 ↔ f onto.

 Pf "⇒":
 Spse f is 1-1.
 Then f: A → ran f is onto by def of ran & def of onto.
 So A ≈ ran f ⊆ A.

 By Corollary 6C, ran f cannot be a proper subset of A.
 So ran f = A. So f is onto.

 "⇐":
 1. Spse f is onto.
   Since A is finite,
 2. Æ bijecton g: A → n for some n ∈ w.
 3. For each a ∈ A, \( f^{-1} \{ \{a\} \} \) ≠ ∅
    since f is onto.
 4. \( g( f^{-1} \{ \{a\} \} ) \) is a nonempty subset of w, by 2 & 3.
 5. Let \( n_a \) be the smallest elem of \( g( f^{-1} \{ \{a\} \} ) \), by well-ord of w.
 6. Let \( x_a ∈ A \) be the element that "corresponds" to \( n_a \), i.e.,

\[ x_a = f^{-1}(n_a). \]

7. Let \( X = \{ x_a | a ∈ A \} \) ⊆ A.

8. Then \( f \upharpoonright X : X → A \)
   is onto since \( \forall a ∈ A, f(x_a) = a. \)

9. \( f \upharpoonright X \) is 1-1 since
   \[ f(x_a) = f(x_b) ⇒ a = b \]
   \[ ⇒ x_a = x_b. \]

10. \( X ≈ A \), by 8 & 9.

11. \( X = A \), by 7, 10, Cor 6C.

12. \( f \upharpoonright X = f \upharpoonright A = f \), by 10.

13. f is 1-1, by 9 & 12. □