Show \( w \) is a trans set.

\textbf{PF} Let \( S(n) \) be the statement \\
\[ n \leq w. \]

1. \( \text{Spse} \ A = \{ n \in w \mid S(n) \text{ is false} \} \neq \emptyset. \)

2. Let \( x \) be the smallest element of \( A \).

3. \( x \neq \emptyset \) since \( S(\emptyset) \) is true vacuously.

4. \( \exists y \in w (y^+ = x) \), by 3.

5. \( y \notin A \) by 2, & b/c \( y < x \) by 4.

6. \( S(y) \) holds, by 5 & 1.

7. \( y \leq w \), by 6.

8. \( \{ y \} \leq w \) since \( y \in w \) (by 4).

9. \( y \cup \{ y \} \leq w \), by 7 & 8 & thm (p.30).

10. \( y^+ \leq w \), by 9 & def of +

11. \( x \leq w \) by 4 & 10.

12. \( S(x) \) holds, by 11 & def of \( S(x) \).

13. \( x \in A \), by 12 & 1.


15. So \( \bot \) is false, \( \text{So } A = \emptyset \).
Show if $A$ has $m$ elements, 
& $B$ has $n$ elements, then
$A \times B$ has $mn$ elements.

Let $f : M \to A$ 
& $g : N \to B$ be bijections.
Define $h : A \times B \to mn$ by
$h(<a, b>) = p^{-1}(a) \cdot n + q^{-1}(b)$
Then $h^{-1} : mn \to A \times B$ is the desired bijection.

P. 89 # 24 Spse:
$m + n = p + q$. Show
$m < p$ iff $n > q$.

Pf: By induction, let $S(m)$
be the statement: "For all $n, p, q \in \mathbb{N}$,
if $m + n = p + q$ & $m < p$, then
$n > q".$ We will show
$S(m)$ holds for all $m \in \mathbb{N}$.
The "iff" then follows by
symmetry. If $m + n = p + q$ and

$q < n$, then $p > q$.

Basis: show $S(0)$ holds.
Spse $0 + n = p + q$, & $0 < p$.
Then $p = k^+$ for some $k$,
so $q + p^+ = n$, so, by
problem 23, $q < n$.

Induction Step: Spse $S(m)$
holds for some $m$.
Let $n, p, q$ be s.t.
$m + n = p + q$, & $m^+ < p$.
Then $m + n^+ = p + q$, since $m^+ + n = m + n^+$
3. Since $m < m^+$ & $m^+ < p$, $m < p$.
4. So $n^+ > q \neq S(m)$ holds &
   by 2.
5. $q < n$, by 4.
6. Case 1: $q < n$. Then we've done.
7. Case 2: $q = n$. Then:
8. $m^+ = p$, by 1 & 7 & cancellation,
9. We have a contradiction by 1 & 8 
   & trichotomy. So only Case 1 holds.