P41. 6. Show that $R$ is a relation iff $R \subseteq \text{dom} A \times \text{ran} A$.

**Def.** A relation is a set of ordered pairs.

**Def.** $x \in \text{dom } R \iff \exists y. (x, y) \in R$.

$x \in \text{ran } R \iff \exists x. (x, y) \in R$.

**Lemma 30.** If $(x, y) \in A$ then $x \in \text{dom } A$ and $y \in \text{ran } A$.

$\Rightarrow$ Suppose $A$ is a relation, need to show $A \subseteq \text{dom } A \times \text{ran } A$.

1. If $x \in A$, $x = (u, v)$ since $A$ is a relation. Def. $\circ$

2. $u \in \text{dom } R$ and $v \in \text{ran } R$ by def dom and ran $\circ$

3. $(u, v) \in \text{dom } A \times \text{ran } A$ by def $x \circ$

4. $A \subseteq \text{dom } A \times \text{ran } A$ by def $\subseteq$.

$\Leftarrow$ Suppose $A \subseteq \text{dom } A \times \text{ran } A$, need to show $A$ is a relation.

1. If $x \in A$, then $x \in \text{dom } A \times \text{ran } A$ by def $\subseteq$.

2. $x \in \{ (u, v) | u \in \text{dom } A \text{ and } v \in \text{ran } A \}$ by def $x$.

3. $x = (u, v)$ for some $u, v$, set. logic $\circ$

4. $A$ is a relation by def of rel. $\circ$

P38. 3. Show $A \times B = \bigcup \{ A \times x \mid x \in B \}$.

$\Rightarrow$ Suppose $x \in A \times B$.

1. $x \in \{ (u, v) | u \in A \text{ and } v \in B \}$ by def $x$.

2. $x = (a, b)$ for some $a, b$, set. logic $\circ$

3. $x \in A \times B$ for some $x \in B$ by def $\subseteq$.

4. $x \in A \times X$ for some $x \in X$, set. logic $\circ$

5. $x \in \{ A \times x \mid x \in B \}$ def $\subseteq$.

$\Leftarrow$ works also
5a. \( A, B \) are sets. Show: \( \exists \) a set \( C \) s.t. for any \( y \in C \Rightarrow y = \{ x \in B \mid \exists x \in A \} \)

Helpful (to me) preamble: What is \( C \)? if \( y \in C \Rightarrow y = \{ x \in B \mid \exists x \in A \} \) for some \( x \in A \). So \( y \) is some set of ordered pairs \( \{u \mid u \in A \} \), but \( u \) can be any member of \( B \).

So \( y \) is a subset of \( A \times B \). (the set of all ordered pairs with a given first member)

What does this make \( C \)? all its members are subsets of \( A \times B \).

So \( C \) must be a subset of \( P(A \times B) \)

**Our task is to show that \( C \) is a set**

\( A, B \) are sets, given \( \exists \) \( A \times B \) is a set, (Lemma 38)

\( \Rightarrow \) \( P(A \times B) \) is a set (Power set axiom)

By the power set axiom:

\( \forall A \forall B \exists C \forall y \in C \iff y \in P(A \times B) \land \exists x \in A \exists y \in B \)

We'll show if \( y \in C \) then \( y \in P(A \times B) \)

<table>
<thead>
<tr>
<th>Step</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.</td>
<td>( y \in C \Rightarrow y = { x \in B \mid \exists x \in A } ) for some ( x \in A ). ( \text{def } C )</td>
</tr>
<tr>
<td>2.</td>
<td>( y \in P(A \times B) ) by ( \exists x \in A ) and ( \exists y \in B ). ( \text{def } P )</td>
</tr>
<tr>
<td>3.</td>
<td>( y \in P(A \times B) ) by ( \text{def } P ), ( \exists )</td>
</tr>
<tr>
<td>4.</td>
<td>( C \subseteq P(A \times B) ) ( \text{def } \subseteq ), ( \forall x \subseteq )</td>
</tr>
</tbody>
</table>

So \( C \subseteq P(A \times B) \), so by Subset Axiom \( C \) is a set.