MANY, MANY MORE INTRINSICALLY KNOTTED GRAPHS

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Abstract. We list more than 200 new examples of minor minimal intrinsically knotted graphs and describe many more that are intrinsically knotted and likely minor minimal.

Introduction

In the early 1980s Conway and Gordon [3] showed that every embedding of K_7, the complete graph on seven vertices, in S^3 contains a nontrivial knot. A graph with this property is said to be **intrinsically knotted** (IK). The question "Which graphs are IK?" has remained open for the past 30 years.

A graph H is a **minor** of another graph G if H can be obtained from a subgraph of G by contracting zero or more edges. A graph G with a given property is said to be **minor minimal** with respect to that property if no proper minor of G has the property. It is easy to show that a graph is IK iff it contains a minor that is minor minimal intrinsically knotted (MMIK). Robertson and Seymour's Graph Minor Theorem [15] says that in any infinite set of graphs, at least one is a minor of another. It follows that for any property whatsoever, there are only finitely many graphs that are minor minimal with respect to that property. In particular, there are only finitely many MMIK graphs. Furthermore, deciding whether one graph is a minor of another can be done algorithmically. Hence, if we knew the finite set of all MMIK graphs, we would be able to decide whether or not any given graph is IK. However, obtaining this finite set, or even putting an upper bound on its size, has turned out to be very difficult. In contrast, Robertson, Seymour, and Thomas [16] settled the corresponding question for **intrinsically linked** (IL) graphs — i.e., graphs for which every embedding in S^3 contains a nontrivial link — in 1995: there are exactly seven MMIL graphs; they are obtained from K_6 by ∇Y and Y∇ moves (we will define these shortly).

Prior to this work 41 MMIK graphs were known. We have found 222 new MMIK graphs, as well as many more IK graphs that are likely minor minimal. In this paper we describe these 222 graphs. For 101 of them we give a "traditional" proof that they are IK. To prove that the remainder are also IK, we rely on the computer program of [12]. The program proves that a graph is IK by showing every embedding of the graph contains a D_4 minor with opposite cycles linked; this is explained in greater detail in Section 4.1. We also prove that all 222 graphs are minor minimal.

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First, some more definitions and terminology. A **spatial** graph is a graph embedded in S^3. A spatial graph is said to be **knotted** (resp., **linked**) if it contains a nontrivial knot (resp., nontrivial link). An abstract graph G is n**-apex** if one can remove n vertices from G to obtain a planar graph. For an edge e of G, $G - e$ denotes the graph obtained by removing e from G and G/e the graph obtained by contracting e.

A ∇Y **move** on an abstract graph consists of removing the edges of a triangle (i.e., 3-cycle) abc in the graph, then adding a new vertex v and connecting it to each of the vertices a, b, and c, as shown in Figure 1. The reverse of this operation is called a Y∇ **move**. Note that in a Y∇ move, the vertex v cannot have degree greater than three. (There is various terminology for this in the literature: ∇ = triangle = Delta = Δ; Y = wye = star; move = exchange = transformation.)

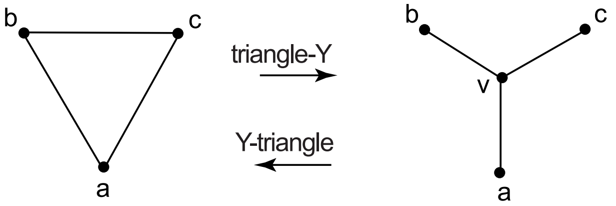

FIGURE 1. ∇Y and Y∇ moves.

If a graph G' is obtained from a graph G by exactly one ∇Y move, we say G' is a **child** of G, and G is a **parent** of G'. A graph that has no degree three vertices can have no parents and we call such a graph **parentless**; a triangle–free graph has no children and is **childless**. If G' is obtained from G by one or more ∇Y moves, we say G' is a **descendant** of G, and G is an **ancestor** of G'. If G' is obtained from G by *zero* or more operations, each of which is a ∇Y or Y∇ move, we say G and G' are **cousins** of each other (thus, being cousins is an equivalence relation). The set of all cousins of G is called the G **family**.

Sachs [17] observed that every child of an IL graph is IL. Essentially the same argument shows that every child of an IK graph is IK. As a corollary of [16], we also know that every parent of an IL graph is IL. In contrast, it is shown in [5] that a parent of an IK graph need not be IK. In this paper we also use the following lemma:

Lemma 1. [2, 14] *If an IK graph G has a MMIK child, then G is MMIK.*

In addition, we make frequent use of a lemma that is a consequence of the observation (due, independently, to [1] and [14]) that the join, $H * K_2$, of H and K_2 is IK if and only if H is nonplanar.

Lemma 2. [1, 14] *If G is 2–apex, then G is not IK.*

The graphs we study here fall into several families. Below we give a quick overview of our results, which are summarized in Table 1; details are provided in the following sections, with one section devoted to each family. Some of these families contain a large number of graphs; we used a computer program to construct these families. The K_7 family consists of 20 graphs, 14 of which were previously known to be MMIK. We show the remaining 6 are not IK (this was also shown,

Family	Graphs (Total)	IK Graphs	MMIK Graphs Known	New
K_7	20	14	14	0
$K_{3,3,1,1}$	58	58	26	32
$E_9 + e$	110	110	0	33
$G_{9,28}$	1609	1609	0	156
$G_{14,25}$	$> 600{,}000$	unknown	0	1

TABLE 1. Families of the 222 new MMIK graphs.

independently, in [10]). The $K_{3,3,1,1}$ family consists of 58 graphs, 26 of which were previously known to be MMIK. We show the remaining 32 are also MMIK. The $E_9 + e$ family consists of 110 graphs. We show that all are IK and exactly 33 of them are MMIK. The $G_{9,28}$ family consists of 1609 graphs. We show they are all IK and at least 156 of them are MMIK. For 101 of these 156 graphs, we prove the graph is MMIK without making use of the computer program of [12]. Sampling results obtained by computer suggest that well over half of the graphs in this family are MMIK. The $G_{14,25}$ family consists of over 600,000 graphs; we don't know the exact number. We only show that $G_{14,25}$ itself is MMIK.

Note that in each family all graphs have the same number of edges since ∇Y and Y∇ moves do not change the number of edges in a graph. However, if two edges of a Y are part of a triangle, then a Y∇ move on that Y results in double edges (i.e., two edges with the same endpoints); in this case we say that the initial graph has a $\overline{\mathrm{Y}}$. It turns out that there is no graph with a $\overline{\mathrm{Y}}$ in the families of each of the graphs K_7, $K_{3,3,1,1}$, E_9+e, $G_{9,28}$, and $G_{13,30}$. (This last graph is described below.) The $G_{14,25}$ family, however, does contain graphs with a $\overline{\mathrm{Y}}$. Whenever our computer program that generates these families encounters a $\overline{\mathrm{Y}}$, it does not perform a Y∇ move on that $\overline{\mathrm{Y}}$, since the resulting graph, after deleting one of its double edges, would have fewer edges than the initial graph. (We prefer to consider graphs with different number of edges to be in distinct families.)

Note that a graph obtained by performing a Y∇ move on a $\overline{\mathrm{Y}}$ followed by deleting one of the resulting double edges can also be obtained by just contracting one of the edges in the $\overline{\mathrm{Y}}$. So it might be interesting to perform such Y∇ moves and study the resulting graphs; they might lead to new MMIK graphs:

Question 3. *Find an example of a MMIK graph that results from contracting an edge of a* $\overline{\mathrm{Y}}$ *in the family of some other MMIK graph.*

In particular, this would be a way to move from the family of one MMIK graph to that of another. We will not pursue this further here as our examples of a $\overline{\mathrm{Y}}$ are in the $G_{14,25}$ family, which is already huge even without considering additional graphs constructed in this way.

Although verifying that a given graph is MMIK can be laborious, using our computer program to generate new candidates for MMIK graphs turned out to be relatively quick. Considering the ease with which we found families of new MMIK graphs, we expect there are many more such families. Since we know of no upper bound on the number of MMIK graphs, it seems that until there is more progress in the theory, it may not be worthwhile to continue the search for more MMIK graphs.

Instead, we propose a couple of questions regarding the size of a MMIK family. The $G_{14,25}$ example shows that these families can become quite large. However, the question of the smallest family remains open. In particular, we can ask:

Question 4. *Is there a MMIK graph that is its own family?*

Such a graph would be both childless and parentless. One way to approach this might be to investigate the following.

Question 5. *Given an arbitrary graph, is there an efficient way of finding, or at least estimating, how many cousins it has?*

For example, the MMIK graph described by Foisy in [7], which has 13 vertices and 30 edges ($G_{13,30}$), has more edges than any of the graphs mentioned above. So we were surprised to learn that its family consists of only seven graphs, making it the smallest family known to us that contains a MMIK graph.

Finally, we remark that our study includes a description of four new MMIK graphs on nine vertices: $E_9 + e$, $G_{9,28}$, and Cousins 12 and 41 of the $K_{3,3,1,1}$ family. A computer search [13] suggests that these, along with the known (i.e., as in [9]) MMIK graphs in the K_7 and $K_{3,3,1,1}$ families, form a complete list of MMIK graphs on nine or fewer vertices. In particular, we expect that the families described in this paper include all MMIK graphs with at most nine vertices.

1. THE K_7 FAMILY

Figure 2 shows the family of 20 graphs derived from K_7 by Y∇ and ∇Y moves. An edge list for each of these 20 graphs can be found in the Appendix [8]. Graphs at the same horizontal level have the same number of vertices, beginning with K_7 (Cousin 1) at the top and concluding with a 14–vertex graph, C_{14} (Cousin 18), at bottom. Edges join parent to child. The numbering of the cousins is somewhat arbitrary: it reflects the order in which these graphs were constructed via ∇Y and Y∇ moves by our algorithm. Note that Cousin 9 is labeled E_9 in [11], and Cousins 16 and 20 are labeled G_6 and G_7 in [5].

Kohara and Suzuki [9] earlier described K_7 and its 13 descendants. None of the six remaining cousins, 9, 14, 16, 17, 19, and 20 are IK. This follows as Cousins 17 and 19 have unknotted embeddings, as shown in Figure 3, and Cousins 9, 14, 16, 20 are ancestors of Cousins 17 and 19. (The unknotted embeddings of Figure 3 were derived from the unknotted embedding of Cousin 20 that appears as Figure 2 in [5].) Thus, the K_7 family yields no new examples of MMIK graphs. This has also been shown, independently, by Hanaki, Nikkuni, Taniyama, and Yamazaki [10].

We remark that E_9 (Cousin 9), a graph on nine vertices and 21 edges, is the smallest graph that is not IK but has an IK child. Indeed, it follows from [11] that descendants of a non-IK graph on fewer edges or fewer vertices would be 2-apex and, therefore, not IK by Lemma 2. Descendants of a graph on 20 or fewer edges also have 20 or fewer edges and, so, are 2-apex. As for graphs on eight vertices, the non-IK examples with 21 or more edges are all subgraphs of two graphs, G_1 and G_2, on eight vertices and 25 edges (see [11, Figure 7]). As all descendants of these two graphs are 2-apex, the same is true of descendants of any subgraphs of G_1 or G_2.

It turns out that by adding one edge to E_9 one can obtain a MMIK graph; we call this graph $E_9 + e$ and describe its family in Section 3.

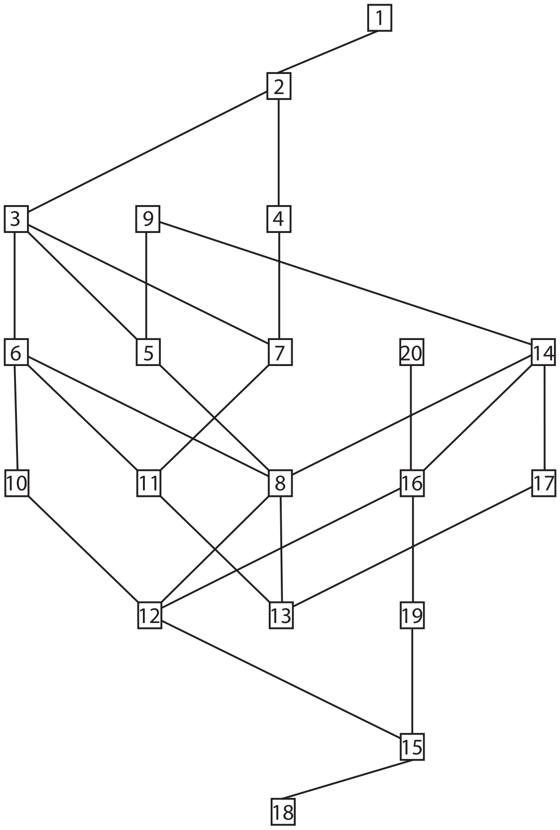

FIGURE 2. The K_7 family.

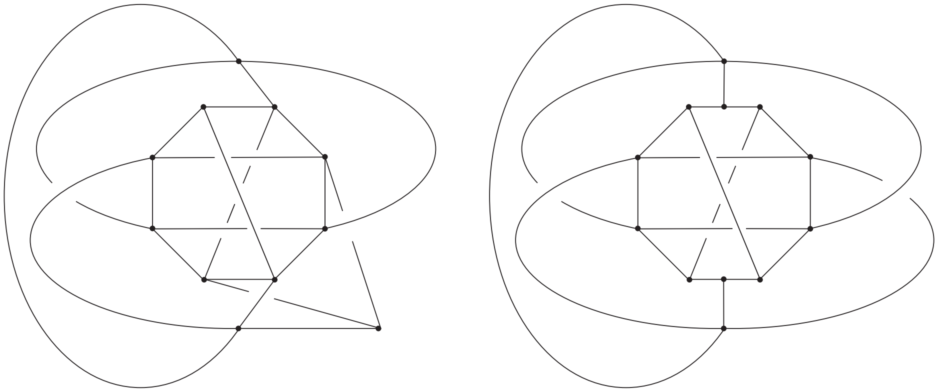

FIGURE 3. Unknotted embeddings of Cousin 17 (left) and Cousin 19 (right) of K_7.

2. THE $K_{3,3,1,1}$ FAMILY

Figure 4 (produced using Mathematica) shows the 58 graphs derived from $K_{3,3,1,1}$ by ∇Y and Y∇ moves. Edge lists for these graphs can be found in the Appendix [8]. The graphs range from the 8 vertex graph $K_{3,3,1,1}$ (Cousin 1) through the

14 vertex graph Cousin 42 (called R_1 in [14]). Kohara and Suzuki [9] described the graph $K_{3,3,1,1}$ and its 25 descendants. These 26 graphs were already known to be MMIK [6, 9]. As we will now show, the remaining 32 graphs in the family are also MMIK.

Proposition 6. *The 58 graphs in the $K_{3,3,1,1}$ family are all MMIK.*

Proof. We first observe that all graphs in the family are IK. For this, it suffices to show that the four parentless cousins, 1, 12, 41, and 58, are intrinsically knotted. Foisy [6] proved this for Cousin 1, $K_{3,3,1,1}$. We handle the remaining three graphs by using the computer program described in [12] to verify that in every embedding of the graph there is a D_4 minor that contains a knotted Hamiltonian cycle.

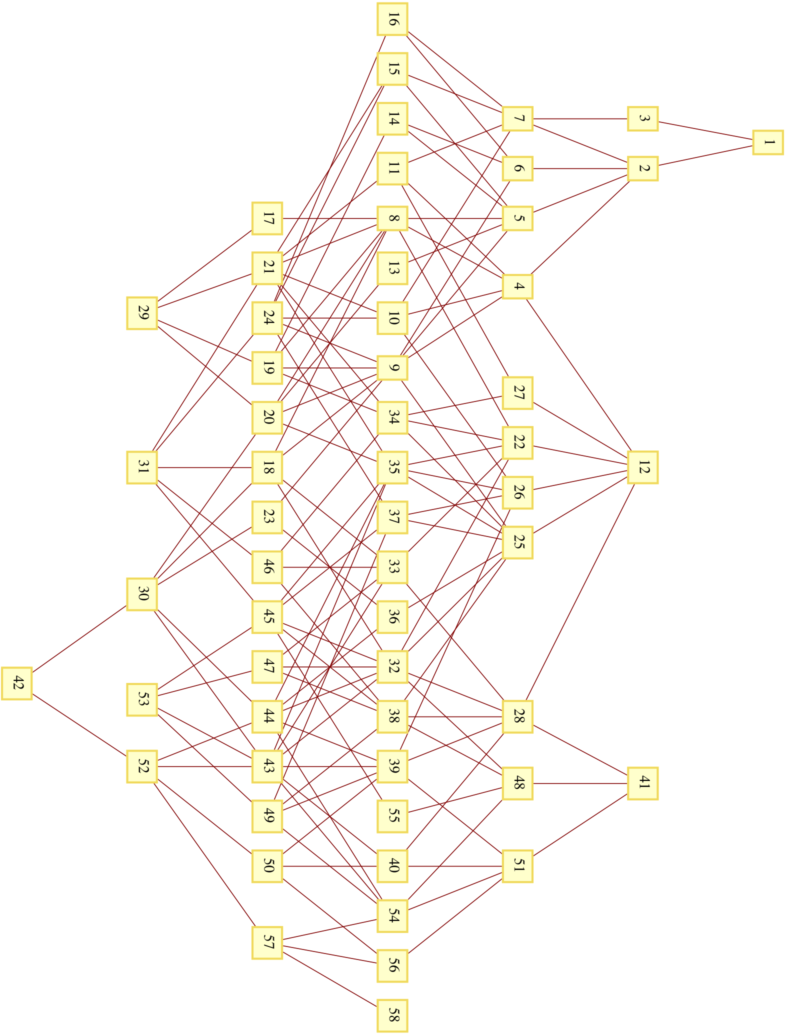

FIGURE 4. The $K_{3,3,1,1}$ family.

Having established that all graphs in the family are IK, by Lemma 1, we can conclude that they are all MMIK once we've shown this for the four childless cousins, 29, 31, 42, and 53. We do know that descendants of $K_{3,3,1,1}$ are MMIK. This combines work of Kohara and Suzuki [9] (who argued that, if $K_{3,3,1,1}$ is MMIK, then all of its descendants are too) and Foisy [6] (who proved that $K_{3,3,1,1}$ is MMIK). As cousins 29, 31, and 42 have $K_{3,3,1,1}$ as an ancestor, the following lemma completes the argument. □

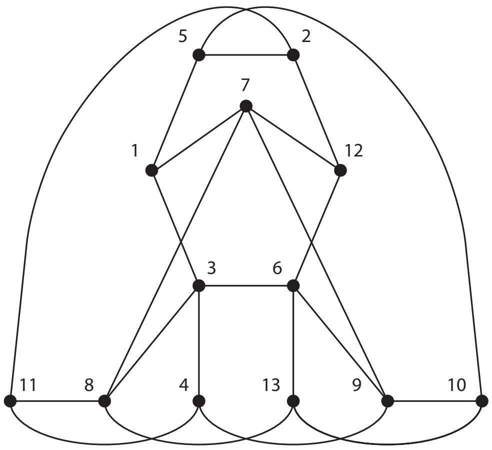

FIGURE 5. Cousin 53 of the $K_{3,3,1,1}$ family.

Lemma 7. *Cousin 53 (Figure 5) of the $K_{3,3,1,1}$ family is MMIK.*

Proof. Let G denote Cousin 53. As in the proof above, all graphs in the family are IK, including G. Since G has no isolated vertices, it will be enough to show that $G - e$ and G/e have knotless embeddings for every edge e in G.

As in Figure 5, the graph has an involution $(1, 12)(2, 5)(3, 6)(4, 13)(8, 9)(10, 11)$. This allows us to identify the 22 edges in pairs with the exception of the edges $(2, 5)$ and $(3, 6)$ (which are fixed by the involution). Thus, up to symmetry, there are 12 choices for the edge e and 24 minors ($G - e$ or G/e) to investigate.

The argument is based primarily on the embedding of G shown in Figure 6, for which there is a single knotted cycle $(1, 5, 2, 12, 6, 3, 4, 11, 8, 13, 10, 9, 7, 1)$, as well as four crossings, labeled A, B, C, D in the figure. By flipping (i.e., interchanging the over- and undercrossing arcs) at selected crossings, we construct two additional embeddings, each having a unique knotted cycle. Let's call the representation of G shown in the figure Embedding 1. If we flip the crossing A, we have what we will call Embedding 2 whose unique knotted cycle is $(1, 3, 6, 12, 2, 11, 4, 9, 7, 8, 13, 10, 5, 1)$. For Embedding 3, we flip the crossings A and C, which gives the knotted cycle $(1, 3, 6, 13, 8, 11, 4, 9, 7, 12, 2, 5, 1)$.

Out of the 12 choices for an edge e, all but one occurs as an edge either in the knotted cycle of Embedding 1 or else in that of Embedding 2. For each such e, this gives an unknotted embedding of $G - e$. The remaining possibility is $e = (6, 9)$ (or, equivalently, $(3, 8)$). In this case, deleting vertices 2 and 3 from $G - (6, 9)$ results in a planar graph (this is not obvious from Figures 5 or 6, but is easy to verify

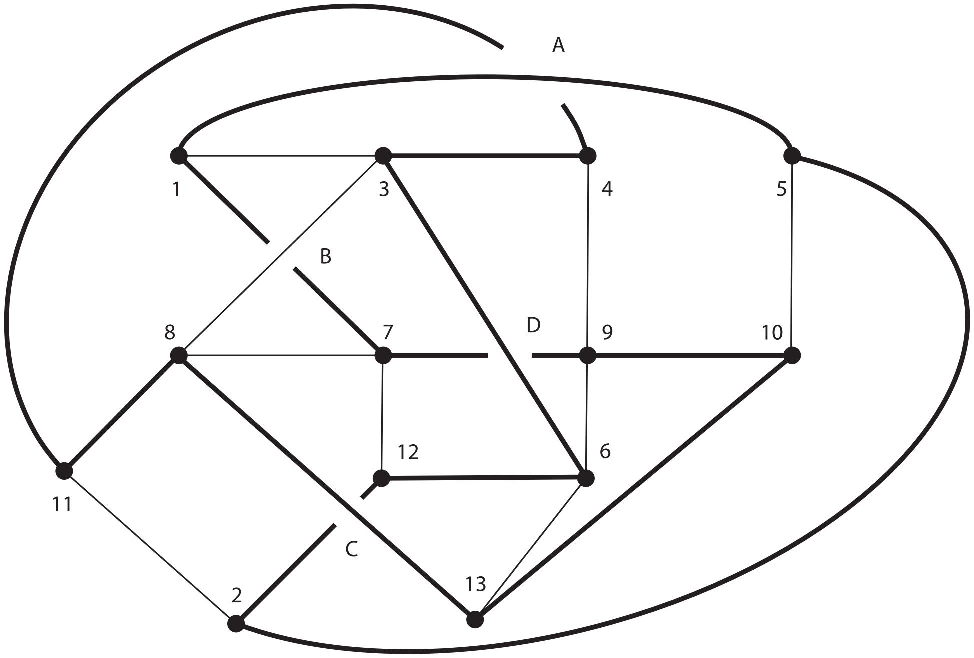

Figure 6. An embedding of Cousin 53 which has a unique knotted cycle (in bold).

manually or using Mathematica). Therefore, $G - (6, 9)$ is 2–apex and, by Lemma 2, not IK. Thus, no minor of the form $G - e$ is IK.

If we contract the edge $e = (1, 3)$ in Embedding 1 (shown in Figure 6), the single knotted cycle becomes two cycles that share the new vertex formed by identifying vertices 1 and 3. Since these two cycles are unknots, this is an unknotted embedding of G/e. Similarly, contracting either edge $e = (4, 9)$ or $e = (5, 10)$ in Embedding 1 leads to an unknotted embedding of G/e. Embedding 2 shows that G/e is unknotted for $e = (1, 7)$, $(2, 5)$, $(6, 9)$, and $(8, 11)$, while Embedding 3 does for $e = (3, 4)$ and $(7, 8)$.

For each of the remaining three choices of e, we give vertices that, when deleted from G/e, yield a planar graph, showing that G/e is 2–apex: for $e = (1, 5)$, delete vertices 4 and 6; for $e = (3, 6)$, delete vertices 2 and 7; for $e = (4, 11)$, delete vertices 5 and 6. Thus, by Lemma 2, none of these graphs is IK, completing the argument for the G/e minors.

As no $G - e$ nor G/e minor is IK, we conclude that G is MMIK. □

3. The $E_9 + e$ family

The graph $E_9 + e$ (see Figure 7) has nine vertices and 22 edges and is formed by adding the edge $(3, 9)$ to E_9. The $E_9 + e$ family consists of 110 cousins; due to its large size, we do not provide here a diagram for the entire family, but only a partial diagram, as explained further below. Edge lists for all 110 cousins, as well as a diagram of the entire family, can be found in the Appendix [8]. The family includes two 8-vertex graphs: the graph whose complement consists of two stars, each on four vertices (see Figure 4vi of [4]) and $K_{3,2,1,1,1} - (b_1, c), (b_1, d)$, whose complement is a triangle and a star of four vertices. We refer the reader to [4] for an explanation of this notation along with a proof that these two graphs are IK (also proved, independently, in [1]). Note that neither is MMIK. The two star graph has K_7 as a minor while $K_{3,2,1,1,1} - (b_1, c), (b_1, d)$ has H_8 (the graph obtained by a ∇Y move on K_7) as a minor.

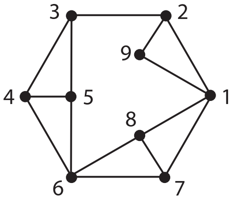

FIGURE 7. The complement of the graph $E_9 + e$.

The family also includes three other parentless graphs, all on ten vertices, which we call Cousins 41, 47, and 50. We can describe these graphs by listing their edges:
Cousin 41: $(1,3),(1,4),(1,5),(1,6),(2,5),(2,6),(2,8),(2,10),(3,6),(3,7),(3,8),(3,9),(4,8),(4,9),(4,10),(5,7),(5,9),(6,9),(6,10),(7,9),(7,10),(8,9)$.
Cousin 47: $(1,3),(1,4),(1,5),(1,6),(2,5),(2,6),(2,8),(2,10),(3,6),(3,7),(3,8),(4,8),(4,9),(4,10),(5,7),(5,8),(5,9),(5,10),(6,9),(7,9),(7,10),(8,9)$.
Cousin 50: $(1,3),(1,4),(1,5),(1,6),(1,10),(2,5),(2,6),(2,8),(2,10),(3,6),(3,7),(3,8),(3,9),(4,8),(4,9),(4,10),(5,7),(5,8),(5,9),(6,9),(7,9),(7,10)$.

To show that all graphs in the $E_9 + e$ family are IK, it's enough to check that all the parentless graphs in the family are IK. We've explained why the two 8-vertex parentless graphs are IK. The program of [12] shows that the four other parentless graphs are IK.

Of the 110 graphs in the family, only 33 are MMIK; they are shown in Figure 8. These 33 graphs are the ancestors of Cousins 43, 46, 83, and 98; they include $E_9 + e$ (Cousin 1) as well as the three parentless 10-vertex graphs, Cousins 41, 47, and 50. The other graphs in the family are all descendants of the two 8-vertex graphs. As the 8-vertex graphs are not MMIK, it follows, by Lemma 1, that the remaining 77 graphs in the family are not MMIK.

The following lemma shows that Cousin 83 and, hence, its 28 ancestors are MMIK. We omit the similar arguments which show that Cousins 43, 46, and 98 are also MMIK.

Lemma 8. *Cousin 83 (Figure 9) of the $E_9 + e$ family is MMIK.*

Proof. The argument is similar to that of Lemma 7 so we will omit some of the details. Let G denote Cousin 83. As G has no symmetries, the 44 minors obtained by removing or contracting each of the 22 edges are pairwise non-isomorphic. We will demonstrate that none of the 44 graphs $G - e$, G/e are IK. Figure 9 shows Embedding 1 with its unique knotted cycle $(1,3,8,13,9,12,6,2,11,5,7,10,4,1)$. By flipping crossings we obtain four other embeddings, each with a unique knotted cycle:
Embedding 2 (flip B): $(1,4,13,9,12,3,8,11,2,10,7,5,1)$;
Embedding 3 (flip A & B): $(1,5,11,2,10,7,3,8,13,9,12,6,1)$;

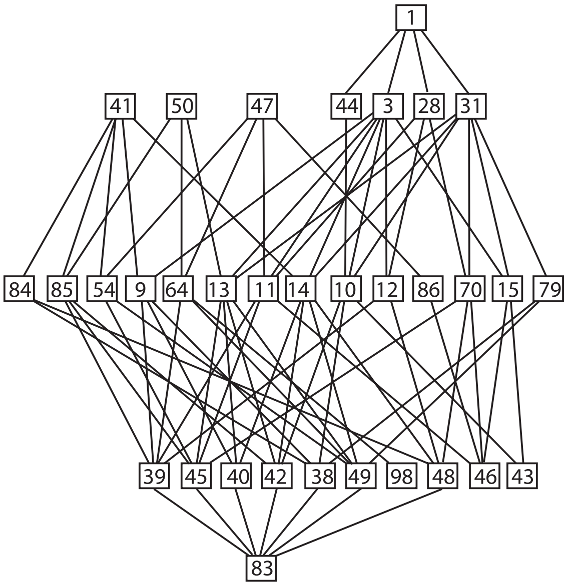

Figure 8. The MMIK cousins of $E_9 + e$.

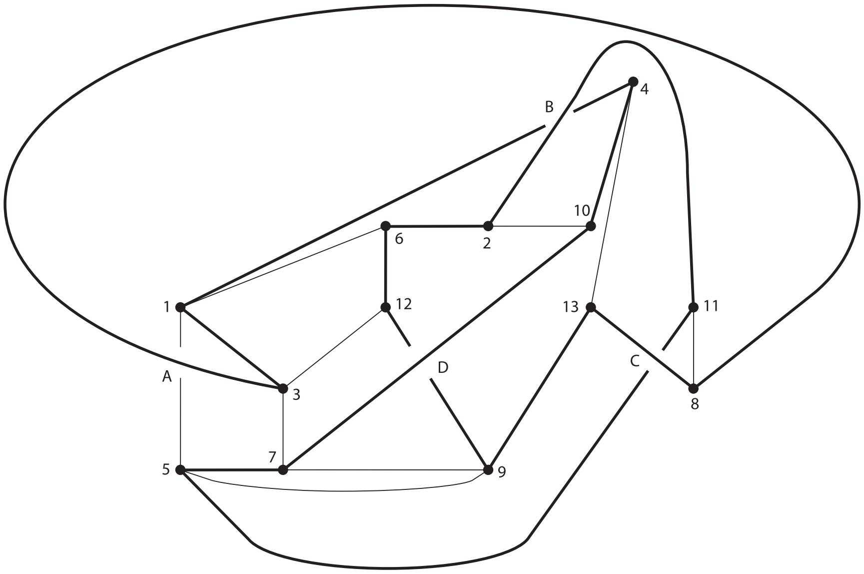

Figure 9. An embedding of Cousin 83.

Embedding 4 (flip C): $(1, 4, 10, 7, 3, 8, 13, 9, 12, 6, 2, 11, 5, 1)$;
Embedding 5 (flip D): $(1, 4, 13, 8, 3, 7, 10, 2, 11, 5, 9, 12, 6, 1)$.

All but one edge e appears in one of the five cycles. The corresponding embedding shows that $G - e$ is not IK. For the remaining edge $e = (7, 9)$, note that removing vertices 2 and 3 from $G - e$ results in a planar graph. So, by Lemma 2, $G - (7, 9)$ is not IK. This completes the argument that no minor of the form $G - e$ is IK.

For 13 of the 22 edges, contracting the edge e turns the unique cycle in at least one of the five embeddings into two unknotted cycles, showing that G/e is not IK. For eight of the remaining nine edges, (namely $(1, 4)$, $(2, 11)$, $(3, 8)$, $(3, 12)$, $(5, 11)$, $(7, 9)$, $(7, 10)$, and $(9, 12)$) G/e is 2–apex and, therefore, not IK. Finally, since $G/(6, 12)$ is isomorphic to Cousin 19 of the K_7 family, it too is not IK. This completes the argument for minors of the form G/e and the proof of the lemma. □

Remark. Although it preserves IKness, the ∇Y move doesn't necessarily preserve MMIKness. Indeed, Cousin 83, which is MMIK by Lemma 8, has a nonMMIK child, Cousin 87. (As a descendant of both of the nonMMIK 8-vertex graphs in this family, Cousin 87 is also nonMMIK by Lemma 1; an edge list for Cousin 87 can be found in our Appendix [8]).

4. The $G_{9,28}$ family

The graph $G_{9,28}$ has nine vertices and 28 edges. It's most easily described in terms of its complement, which is the disjoint union of a 7–cycle on the vertices $1, 2, \dots, 7$ and the edge $(8, 9)$.

The $G_{9,28}$ family, listed fully in [8], consists of 1609 cousins, 25 of which, including $G_{9,28}$ itself, are parentless. The remaining cousins are descendants of one or more of these 25 parentless graphs. We used the computer program of [12] to verify that each of these 25 parentless graphs is IK; hence all 1609 cousins are IK. Here we give a "traditional proof" that $G_{9,28}$ is IK. We note that $G_{9,28}$ and its descendants account for 1062 of these 1609 cousins; thus these 1062 graphs are IK even without the "computer proof."

We also show in this section that Cousin 1151 of $G_{9,28}$, which is a descendant of $G_{9,28}$, is MMIK. Cousin 1151 and its ancestors form a set of 156 graphs (only 101 of which are $G_{9,28}$ or its descendants). Thus all of these 156 graphs are MMIK.

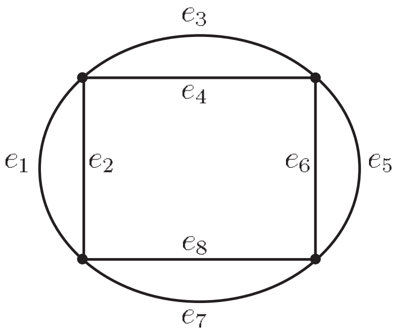

Figure 10. The graph D_4.

4.1. $G_{9,28}$ **is IK.** In this subsection, we show that $G_{9,28}$ is IK. We'll make use of a lemma due independently to Foisy [6] and Taniyama and Yasuhara [18], which we restate here. Figure 10 shows the multigraph D_4. For each $i = 1, 2, 3, 4$, let C_i denote the cycle consisting of the two edges e_{2i-1} and e_{2i}. For any given embedding of D_4, let σ denote the mod 2 sum of the Arf invariants of the 16 Hamiltonian cycles in that embedding of D_4. ($\mathrm{Arf}(K)$ equals the reduction modulo 2 of the second coefficient of the Conway polynomial of K.) Since the unknot has Arf invariant zero, if $\sigma \neq 0$ there must be a nontrivial knot in the embedding. The lemma shows that this will happen whenever the mod 2 linking numbers, $\mathrm{lk}(C_i, C_j)$, of both pairs of opposing cycles are non-zero.

Lemma 9. [6, 18] *Given an embedding of the graph D_4, $\sigma \neq 0$ if and only if $lk(C_1, C_3) \neq 0$ and $lk(C_2, C_4) \neq 0$.*

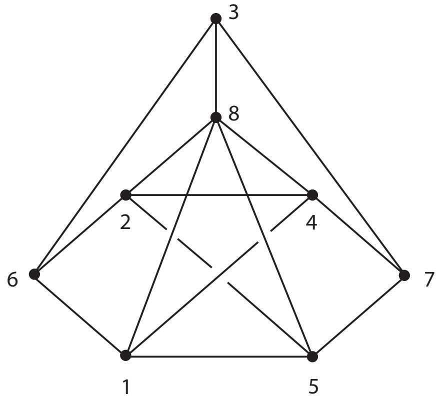

FIGURE 11. The Petersen graph P_8 is realized as a subgraph, S_1, of $G_{9,28}$.

Proposition 10. *The graph $G_{9,28}$ is IK.*

Proof. First observe that the Petersen graph P_8 shown in Figure 11 is a subgraph of $G_{9,28}$ since the complement of P_8 contains the 7-cycle $(1, 2, 3, 4, 5, 6, 7)$ and the edge $(8, 9)$; we'll call this subgraph S_1. By cyclically permuting the vertex labels $1, 2, \ldots, 7$ in Figure 11, we obtain six more subgraphs, $S_2, S_3, \ldots S_7$, of $G_{9,28}$, each isomorphic to P_8. There are eight pairs of cycles in each S_i. For example, the eight links in S_1 and S_2 are:

i	l_{i1}	l_{i2}	l_{i3}	l_{i4}
1	148, 36257	158, 36247	248, 36157	258, 36147
2	258, 47361	268, 47351	358, 47261	368, 47251

i	l_{i5}	l_{i6}	l_{i7}	l_{i8}
1	2475, 3618	5162, 3748	1475, 3628	4261, 3758
2	3516, 4728	6273, 4158	2516, 4738	5372, 4168

In the table, we've listed the indices of the vertices in each cycle. Thus l_{11}, S_1's first link, consists of the cycles $(1, 4, 8, 1)$ and $(3, 6, 2, 5, 7, 3)$. We will frequently use this abbreviated notation in what follows. Note that each link l_{2j} can be obtained from the one above it, l_{1j}, by applying the cyclic permutation $\gamma = (1, 2, 3, 4, 5, 6, 7)$; we'll write $l_{2j} = \gamma(l_{1j})$. In a similar way, we determine the links l_{ij} for each $i = 3, 4, 5, 6, 7$ by repeatedly applying γ.

Fix an arbitrary embedding of $G_{9,28}$. We wish to show that there is a knotted cycle in that embedding. We'll argue that $G_{9,28}$ has a D_4 minor embedded with opposite cycles linked. Using Lemma 9, this implies there is a knotted cycle in the D_4 and we will refer to such a D_4 minor as a "knotted D_4." We can then identify the knot in the D_4 with a knotted cycle in the given embedding of $G_{9,28}$.

As shown by Sachs [17], in any embedding of the Petersen graph P_8, the mod 2 sum of the linking number over the eight pairs of cycles is non-zero. This means that, in each S_i, at least one pair of cycles l_{ij} has non-zero linking number mod 2. To simplify the exposition, for the remainder of this proof, we will use "linked" to mean "has nonzero linking number mod 2."

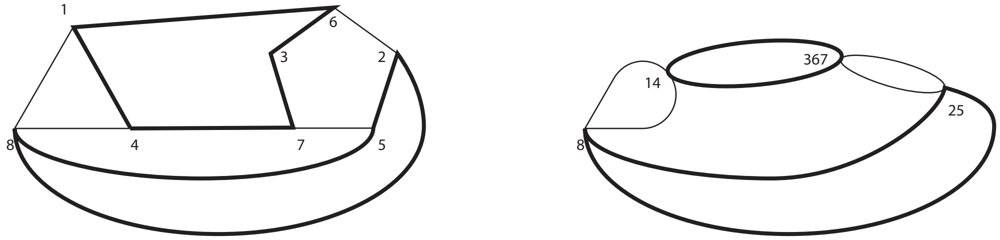

Figure 12. Contracting $(1,4)$, $(2,5)$, $(3,6)$, and $(3,7)$ results in a D_4.

Suppose first that it's l_{11} that is linked. We'll use this to deduce that l_{26} or l_{28} is linked. We'll denote this situation by writing "$l_{11} \Rightarrow l_{26}$ or l_{28}." We will argue that any other l_{2j}, if linked, would result in a knotted D_4 minor and therefore a knotted cycle in the given embedding of $G_{9,28}$. If l_{21} is linked, then contracting the edges $(1,4)$, $(2,5)$, $(3,6)$, and $(3,7)$ results in a D_4 graph with one set of opposing cycles arising from l_{11} and the other from l_{21} (see Figure 12). Assuming both pairs are linked, this results in a nontrivial knot in the D_4 graph by Lemma 9 and hence a nontrivial knot in the embedding of $G_{9,28}$. So, we can assume l_{21} is not linked. Similarly, if l_{27} is linked, contracting $(2,5)$, $(2,6)$, $(3,7)$, and $(4,8)$ results in a D_4 with linked opposite cycles arising from l_{11} and l_{27}.

Suppose that among the l_{2j} pairs, it's l_{22} that is linked. Here, we will first decompose the cycles of l_{11} and l_{22}. Since $(3,5)$ is an edge of $G_{9,28}$, in homology, we can think of the cycle 36257 (i.e., $(3,6,2,5,7,3)$) as the sum $[\gamma_2 + \gamma_3]$ of the cycles $\gamma_2 = 3625$ and $\gamma_3 = 357$. The sum is linked with the other component of l_{11}, 148, so we deduce that exactly one of γ_2 and γ_3 is also linked with 148 (see [6, Lemma 3.1]). We will refer to this way of dividing 36257 into 3625 and 357 as "cutting along 35." If it's 3625 that's linked with 148, then contracting $(1,4)$, $(3,5)$, and $(2,6)$ results in a D_4 whose opposite cycles arise from the linked cycles 3625 and 148 and the linked cycles of l_{22}. Consequently, by Lemma 9, the embedding of $G_{9,28}$ has a knotted cycle in this case. If instead it's 357 that's linked with 148, we will need to cut the cycle 47351 of l_{22} along 13. This leaves two cases. If it's 4731 that links the other component of l_{22}, 268, then after contracting $(1,4)$, $(2,5)$, $(2,6)$, and $(3,7)$, we'll have a knotted embedding of D_4. On the other hand, if it's 135 that links 268, we'll want to contract $(1,4)$, $(2,6)$, $(2,7)$ and $(3,5)$ to achieve an embedding of D_4 that implies a knotted cycle in $G_{9,28}$ by Lemma 9.

Similarly, if l_{23} or l_{25} is linked, we'll need to cut 36257 along 35. For l_{24}, we again cut 36257 along 35 and further, cut 47251 along 42. Thus, in every case other than $j = 6$ and $j = 8$, we've shown that assuming l_{11} and l_{2j} are both linked leads to an embedding of D_4 that forces a knotted cycle in our embedding of $G_{9,28}$. This shows that $l_{11} \Rightarrow l_{26}$ or l_{28}.

In much the same way, we now show $l_{11} \Rightarrow l_{53}$ or l_{56}. It's straightforward to verify that there'll be a knotted D_4 in case both l_{11} and one of l_{51}, l_{52}, or l_{57} are linked. As for l_{54}, it's the same link as l_{22}, which we treated above. The two remaining cases require cutting, as we will now describe. If l_{11} and l_{55} are both linked, cut 7358 along 57 and use the edges $(6,9)$ and $(7,9)$ to identify 36257 as the sum (in homology) of 3697 and 62579. (We'll call this operation "cutting along 697.") Finally, if it's l_{58} that is linked, cut 7428 along 27 and 36257 along 35.

As a final step in the argument for l_{11}, we construct a new P_8 subgraph, T_1, from S_1 by interchanging the vertex labels 8 and 9. Thus, the linked pairs in T_1 are $m_{11} = 149, 36257$; $m_{12} = 159, 36247$; …; $m_{18} = 4261, 3759$ (compare with the table of l_{1j} above). Again, by [17], at least one of these m_{1j} is linked in any embedding of $G_{9,28}$. We'll argue that this, together with $l_{11} \Rightarrow l_{26}$ or l_{28} and with $l_{11} \Rightarrow l_{53}$ or l_{56}, imply that there is a knotted D_4 in our embedding of $G_{9,28}$.

First notice that l_{11} will form a D_4 with m_{12}, m_{13}, m_{15}, m_{16}, and m_{18} (after cutting 36257 along 27). In other words, $l_{11} \Rightarrow m_{11}$, m_{14}, or m_{17}. Now, each of the following pairs of link forms a D_4: m_{11} and l_{26} (cut 36257 along 35), m_{14} and l_{26} (no cuts), m_{14} and l_{28} (cut 36147 along 13), and m_{11} and l_{28} (cut 36257 along 85, 86, 87, i.e., $36257 = 3687 + 785 + 5862$). Since $l_{11} \Rightarrow l_{26}$ or l_{28}, we deduce that if l_{11} is linked, then we can assume m_{17} is linked, too, i.e., $l_{11} \Rightarrow m_{17}$.

To complete the argument for l_{11}, we construct T_2, another embedding of P_8, and its links m_{2j}, in the usual way by applying the 7–cycle γ to the vertex indices of T_1. Using the same type of argument as above, we can see that $m_{17} \Rightarrow m_{22}$ or m_{24} (the only case that requires any cuts is when m_{17} is combined with m_{28}, where we cut 3629 along 382 and 5372 along 57). As we showed earlier, $l_{11} \Rightarrow l_{26}$ or l_{28}; and we can show m_{22} combined with either l_{26} or l_{28} yields a D_4 (for m_{22} with l_{26}, cut 47351 along 13; for m_{22} with l_{28}, no cuts); we therefore conclude that if m_{22} and l_{11} are both linked, then there is a knotted D_4. Similarly, we know $l_{11} \Rightarrow l_{53}$ or l_{56}; and we can show m_{24} combined with either l_{53} or l_{56} produces a D_4 (for m_{24} with l_{53}, cut 47251 along 187; for m_{24} with l_{56}, cut 47251 along 24); hence l_{11} and m_{24} give a D_4. It follows that if l_{11} and m_{17} are both linked, there is a nontrivial knot in our embedding of $G_{9,28}$. Thus, when l_{11} is linked, no matter which pair of cycles m_{1j}, $j = 1, \dots, 8$ is linked in the Petersen graph T_1, we will have a knotted cycle in our embedding of $G_{9,28}$. This completes the argument for l_{11}.

Next, we show that for all $i = 1, \dots, 7$, we can assume all l_{ij} except l_{i3}, l_{i5}, and l_{i8} are unlinked. Indeed, we have shown that if l_{11} is linked, there will be a knot in our embedding of $G_{9,28}$, which is the goal of our proof. Thus, we can assume l_{11} is not linked. By symmetry, the same argument can be applied to each l_{i1}, $i = 1, \dots, 7$. Since l_{14} becomes l_{11} after applying the involution $\delta = (1,5)(2,4)(6,7)$, which is a symmetry of $G_{9,28}$, we can likewise assume l_{14}, and hence every l_{i4} is unlinked. Also, $l_{12} = l_{51}$ which, as we have already noted, is not linked. So, we may assume, no l_{i2} is linked. Next, suppose l_{17} is linked. We have mentioned that $m_{17} \Rightarrow m_{22}$ or m_{24}, and, by symmetry, the same argument shows $l_{17} \Rightarrow l_{22}$ or l_{24}. However, as we have noted, no l_{i2} or l_{i4} is linked, or else we will have a knotted D_4. Thus, l_{17} is not linked either. Again, by symmetry, this implies no l_{i7} is linked. Since $l_{16} = \delta(l_{17})$, we can also assume l_{16}, and hence all l_{i6}, are also not linked. Thus, only pairs of the form l_{ij} with $j = 3$, 5, or 8 are linked.

Now, if l_{13} and l_{23} are both linked, we get a knotted D_4 by doing a few cuts at various stages (cut 36157 along 35, 47261 along 42, 4261 along 496, and 3615 along 391). Thus, $l_{13} \Rightarrow l_{25}$ or l_{28}; and, by symmetry, we have

$$l_{i3} \Rightarrow l_{(i+1)5} \text{ or } l_{(i+1)8} \tag{1}$$

where, whenever the first index i of l_{ij} is greater than 7 or less than 1 (which comes up further below), we reduce i mod 7, except that we use 7 instead of 0. We also argue that $l_{15} \Rightarrow l_{53}$, by showing l_{15} and l_{55} together give a D_4 (cut 7358 along 57) and l_{15} with l_{58} together give a D_4 (no cuts). It follows from symmetry that

$$l_{i5} \Rightarrow l_{(i+4)3} \tag{2}$$

We now apply the permutation $\delta = (1,5)(2,4)(6,7)$ to implication (2) above. First, recall that $\gamma^k(l_{ij}) = l_{(i+k)j}$, where, as before, γ is the 7-cycle $(1,2,3,4,5,6,7)$. Also, note that $\delta\gamma = \gamma^{-1}\delta$. Hence, since $\delta(l_{15}) = l_{18}$, we get $\delta(l_{i5}) = \delta\gamma^{i-1}(l_{15}) = (\gamma^{-1})^{i-1}\delta(l_{15}) = \gamma^{1-i}(l_{18}) = l_{(2-i)8}$. Also, $\delta(l_{13}) = l_{13}$, which, by a similar argument as above, gives $\delta(l_{i3}) = l_{(2-i)3}$. Thus, applying δ to implication (2) gives $l_{(2-i)8} \Rightarrow l_{(2-i-4)3}$, which, by replacing both occurrences of $2-i$ with i, gives

$$l_{i8} \Rightarrow l_{(i+3)3} \tag{3}$$

Combining implications (1), (2), and (3) yields $l_{i3} \Rightarrow l_{(i+4)3}$ or $l_{(i+5)3}$. In particular, $l_{13} \Rightarrow l_{53}$ or l_{63}, $l_{53} \Rightarrow l_{23}$ or l_{33}, and $l_{63} \Rightarrow l_{33}$ or l_{43}. These three implications together give $l_{13} \Rightarrow l_{23}$ or l_{33} or l_{43}. We've already seen that l_{13} and l_{23} together give a D_4. We also check that l_{13} and l_{43} give a D_4 (cut 36157 along 697, 61579 along 59, 62413 along 64, 6413 along 61). Thus we conclude that $l_{13} \Rightarrow l_{33}$, which, by symmetry, gives

$$l_{i3} \Rightarrow l_{(i+2)3} \tag{4}$$

Applying implication (4) repeatedly gives $l_{13} \Rightarrow l_{33} \Rightarrow l_{53} \Rightarrow l_{73} \Rightarrow l_{23}$, which means if l_{13} is linked, we get a knotted D_4 (since l_{13} and l_{23} give a D_4). By symmetry, we get a knotted D_4 if any l_{i3} is linked. This, and implications (2) and (3), together imply that if any l_{i5} or l_{i8} is linked, we get a knotted D_4.

This completes the proof that $G_{9,28}$ is IK. No matter which l_{1i} is linked, we have found a nontrivial knot in the given embedding of $G_{9,28}$.

□

4.2. **Cousin 1151 of $G_{9,28}$.** We now focus on the childless Cousin 1151 of $G_{9,28}$ (Figure 13) and show that it is minor minimal. This implies, by Lemma 1, that the 156 graphs consisting of Cousin 1151 and its ancestors are all MMIK.

It turns out that Cousin 1151 has no symmetries; hence we consider all its 56 minors obtained by deleting or contracting each of its 28 edges. Of these 56 minors, 54 are 2–apex. Below, we list each of these 54 graphs as $G-e$ or G/e, followed by the two vertices that can be removed to obtain a planar graph. The remaining two graphs are listed as "not 2–apex". Note that whenever we contract an edge (a,b) in G, we relabel some of the vertices in $G/(a,b)$, as follows: If $a<b$, then we use the label a for the vertex that edge (a,b) contracts to; furthermore, we take the vertex in G with the largest label and relabel it as vertex b in $G/(a,b)$.
$G-(1,8)$, $\{4,5\}$; $G/(1,8)$, $\{1,2\}$; $G-(1,9)$, $\{4,14\}$; $G/(1,9)$, $\{1,2\}$; $G-(1,10)$, $\{2,8\}$; $G/(1,10)$, $\{4,14\}$; $G-(1,11)$, $\{2,3\}$; $G/(1,11)$, $\{5,14\}$; $G-(2,12)$, $\{1,3\}$; $G/(2,12)$, $\{2,8\}$; $G-(2,13)$, $\{1,7\}$; $G/(2,13)$, $\{1,2\}$; $G-(2,14)$, $\{1,5\}$; $G/(2,14)$,

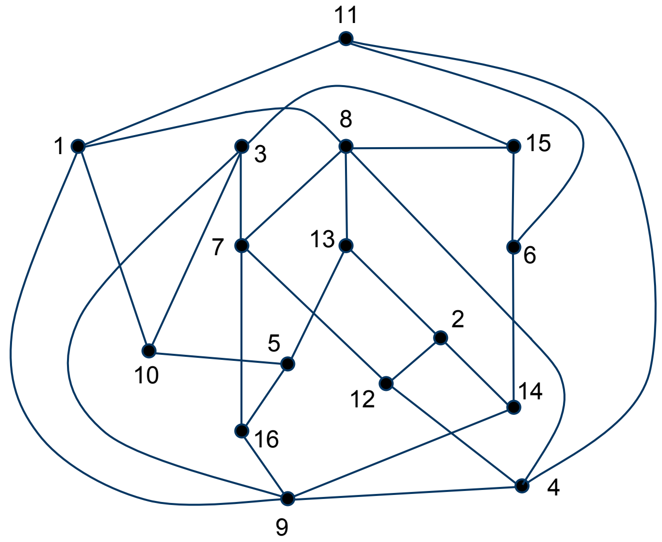

Figure 13. Cousin 1151 of $G_{9,28}$.

$\{2,3\}$; $G-(3,7)$, $\{1,2\}$; $G/(3,7)$, $\{3,4\}$; $G-(3,9)$, $\{6,7\}$; $G/(3,9)$, $\{1,2\}$; $G-(3,10)$, $\{2,4\}$; $G/(3,10)$, $\{3,8\}$; $G-(3,15)$, $\{1,9\}$; $G/(3,15)$, $\{2,3\}$; $G-(4,8)$, $\{7,13\}$; $G/(4,8)$, $\{2,3\}$; $G-(4,9)$, $\{1,7\}$; $G/(4,9)$, $\{2,3\}$; $G-(4,11)$, $\{2,3\}$; $G/(4,11)$, $\{7,13\}$; $G-(4,12)$, $\{2,8\}$; $G/(4,12)$, $\{1,13\}$; $G-(5,10)$, $\{3,8\}$; $G/(5,10)$, not 2–apex; $G-(5,13)$, $\{1,2\}$; $G/(5,13)$, $\{5,6\}$; $G-(5,16)$, $\{1,2\}$; $G/(5,16)$, not 2–apex; $G-(6,11)$, $\{5,7\}$; $G/(6,11)$, $\{2,3\}$; $G-(6,14)$, $\{1,7\}$; $G/(6,14)$, $\{5,6\}$; $G-(6,15)$, $\{1,7\}$; $G/(6,15)$, $\{6,7\}$; $G-(7,8)$, $\{3,4\}$; $G/(7,8)$, $\{1,2\}$; $G-(7,12)$, $\{1,9\}$; $G/(7,12)$, $\{3,14\}$; $G-(7,16)$, $\{3,4\}$; $G/(7,16)$, $\{1,2\}$; $G-(8,13)$, $\{1,2\}$; $G/(8,13)$, $\{3,4\}$; $G-(8,15)$, $\{2,3\}$; $G/(8,15)$, $\{1,9\}$; $G-(9,14)$, $\{2,3\}$; $G/(9,14)$, $\{1,7\}$; $G-(9,16)$, $\{1,2\}$; $G/(9,16)$, $\{6,13\}$.

By Lemma 2, all the 54 graphs that are 2–apex have knotless embeddings. In Figures 14 and 15 we display knotless embeddings for the two graphs that are not 2–apex. (We used a computer program to verify that every cycle in these two embeddings is a trivial knot.)

5. The $G_{14,25}$ Family

The graph $G_{14,25}$, depicted in Figure 16, has 14 vertices and 25 edges: $(1,6)$, $(1,9)$, $(1,10)$, $(1,11)$, $(2,6)$, $(2,7)$, $(2,8)$, $(2,14)$, $(3,10)$, $(3,12)$, $(3,13)$, $(4,6)$, $(4,7)$, $(4,9)$, $(4,11)$, $(5,7)$, $(5,8)$, $(5,10)$, $(5,14)$, $(6,13)$, $(7,12)$, $(8,11)$, $(8,13)$, $(9,12)$, $(9,14)$.

We obtained this graph by starting with one of the cousins of $G_{9,28}$ that is IK but not MM, and repeatedly deleting or contracting edges (a total of three edges) until we arrived at a MMIK graph.

The graph $G_{14,25}$ is interesting since it is a MMIK graph with over 600,000 cousins! We don't know exactly how many cousins it has; we stopped the computer program after about one week of continuous operation, since we had no upper bound

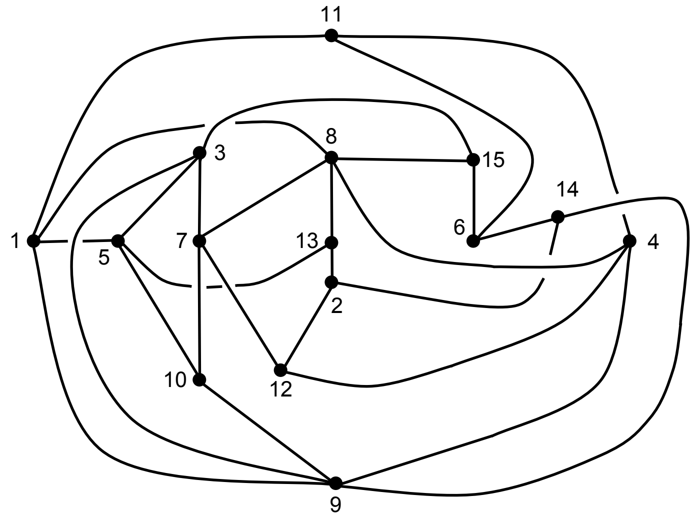

Figure 14. A knotless embedding of the graph obtained by contracting edge $(5, 10)$ in Cousin 1151 of $G_{9,28}$ (vertex 10 used to be vertex 16).

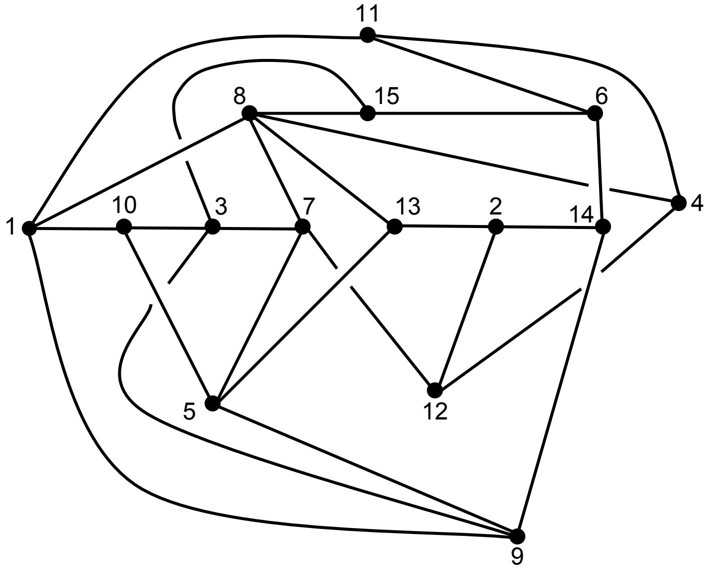

Figure 15. A knotless embedding of the graph obtained by contracting edge $(5, 16)$ in Cousin 1151 of $G_{9,28}$.

on the number of cousins and therefore had no idea how much longer the program might continue to run. We sampled a small number of these cousins, which turned out not to be MMIK. Nevertheless, we wouldn't be surprised if such a large family turned out to contain hundreds or thousands of MMIK graphs.

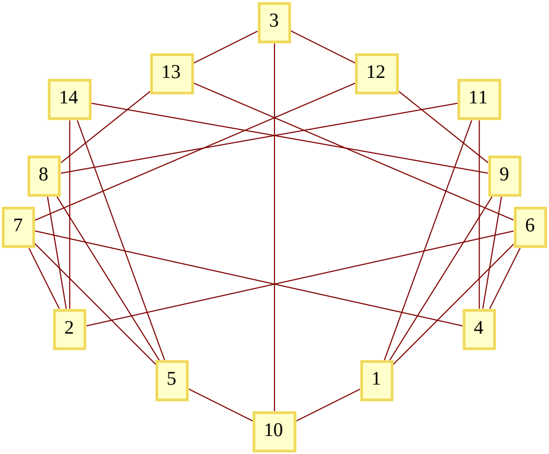

FIGURE 16. The graph $G_{14,25}$.

Lemma 11. *The graph $G_{14,25}$ is MMIK.*

Proof. Let G denote the graph $G_{14,25}$. We show that G is IK by using the computer program described in [12] to verify that there is a D_4 minor with a knotted Hamiltonian cycle in every embedding of the graph. To prove that G is MM, since it has no isolated vertices, it will be enough to show that for every edge e in G, neither $G - e$ nor G/e is IK.

The graph G has an involution $(1,5)(2,4)(6,7)(8,9)(11,14)(12,13)$ which allows us to identify all its 25 edges in pairs, with the exception of the edge $(3,10)$ (which is fixed by the involution). Thus, up to symmetry, there are 13 choices for the edge e and 26 minors ($G - e$ or G/e) to investigate. Each of these 26 minors turns out to be 2–apex. Below, we list each of them as $G - e$ or G/e, followed by the two vertices that can be removed to obtain a planar graph. (See the note in Section 4.2 about vertex relabeling.)
$G-(1,6)$, $\{2,3\}$; $G/(1,6)$, $\{1,6\}$; $G-(1,9)$, $\{2,4\}$; $G/(1,9)$, $\{1,2\}$; $G-(1,10)$, $\{2,4\}$; $G/(1,10)$, $\{2,4\}$; $G-(1,11)$, $\{2,3\}$; $G/(1,11)$, $\{1,7\}$; $G-(2,6)$, $\{1,7\}$; $G/(2,6)$, $\{2,3\}$; $G-(2,7)$, $\{1,3\}$; $G/(2,7)$, $\{1,2\}$; $G-(2,8)$, $\{3,5\}$; $G/(2,8)$, $\{1,3\}$; $G-(2,14)$, $\{1,3\}$; $G/(2,14)$, $\{1,2\}$; $G-(3,10)$, $\{2,4\}$; $G/(3,10)$, $\{2,4\}$; $G-(3,12)$, $\{1,2\}$; $G/(3,12)$, $\{3,4\}$; $G-(6,13)$, $\{1,7\}$; $G/(6,13)$, $\{2,5\}$; $G-(8,11)$, $\{1,7\}$; $G/(8,11)$, $\{2,3\}$; $G-(8,13)$, $\{2,3\}$; $G/(8,13)$, $\{1,7\}$.

It follows from Lemma 2 that each of these 26 minors has a knotless embedding, and hence $G_{14,25}$ is MMIK. □

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