A NOTE ON A SUBTRACTION–TRANSFER GAME

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Received: 1/31/19, Revised: 1/23/20, Accepted: 6/9/20, Published: 6/18/20

Abstract

We discuss a subtraction game on two piles with transfers allowed from one of the piles to the other. We prove results on winning strategies and prove and conjecture periodicity properties of the Grundy values.

1. Introduction

The following problem appeared on the 2014 Bay Area Mathematical Olympiad [1] and it resurfaced in the 2018 fall edition of Emissary [4]. It is about an impartial 2-player subtraction game (cf. [2]) with two piles and transfer allowed from one of the piles to the other. The game does not seem to fit into the usual categories. We note that accordingly to Ferguson [5] "in general, games of this sort, in which the sizes of two or more boxes may change simultaneously in one move, may not be written as a disjunctive sum of games."

"**C and 1.** Amy and Bob play a game. They alternate turns, with Amy going first. At the start of the game, there are 20 cookies on a red plate and 14 on a blue plate. A legal move consists of eating two cookies taken from one plate, or moving one cookie from the red plate to the blue plate (but never from the blue plate to the red plate). The last player to make a legal move wins; in other words, if it is your turn and you cannot make a legal move, you lose, and the other player has won. Which player can guarantee that they win no matter what strategy their opponent chooses? Prove that your answer is correct."

The following clever solution was provided in [1].

"Let's write the number of cookies in the red and blue plate, respectively, as an ordered pair (x, y), so that the legal moves are to $(x - 2, y)$ or $(x, y - 2)$ or $(x - 1, y + 1)$. Thus the only positions with no legal move are (0, 0) and (0, 1), and since cookies are eaten in pairs, the final position is determined by the original number of cookies.

Starting from (20,14), we know that eventually all the cookies will be eaten, so there are exactly $(20 + 14)/2 = 17$ cookie-eating moves. There may also be some number of moves from the first pile to the second pile, but since an even number of cookies are eaten from each pile, there must be an even number of such moves. Thus, the total number of moves in the game is odd, and the first player gets the last legal move.

For general starting positions, there are a few cases to examine depending on whether the total number of cookies is even and the number of cookies in pile 2 is even, but the logic is similar."

This is a combinatorial game which has the features of subtraction games (cf. [2]) with a twist: transfers are allowed from the red plate to the blue one. We determine the Sprague–Grundy or simply Grundy numbers based on the Sprague–Grundy theory (cf. [2]) for various generalizations of this game. The game (a, b, c) corresponds to taking either a cookies from the red plate or b cookies from the blue plate or moving c cookies from the red plate to the blue one.

The transfer game (a, b, c) can be also viewed as the bidimensional vector addition game with $S = \{(-a, 0), (0, -b), (-c, +c)\}$ as addition set; cf. [3]. The originally mentioned game has the addition set $S = \{(-2, 0), (0, +2), (-1, 1)\}$.

We experimentally find, prove, or conjecture some periodic behaviors. We assume that $a, b, c \in \mathbb{Z}^+$ with $\mathbb{Z}^+$ being the set of positive integers. We also use the notation $\mathbb{N}$ for the set of natural numbers and $\mathbb{Z}$ for the set of all integers. Any position of the game will be represented by an ordered pair (x, y) where x and y denote the number of cookies in the red and blue plates, respectively. We denote the initial cookie counts by n_R and n_B on the red and blue plates, respectively. The Grundy value of the game (a, b, c) with initial cookie counts n_R and n_B is denoted by $\mathcal{G}(n_R, n_B) = \mathcal{G}_{a,b,c}(n_R, n_B)$.

2. Main Results

We say that the game (a, b, c) is fully periodic with $p = p(a, b, c)$ and $q = q(a, b, c)$ if $\mathcal{G}_{a,b,c}(n_R + p, n_B) = \mathcal{G}_{a,b,c}(n_R, n_B)$ and $\mathcal{G}_{a,b,c}(n_R, n_B + q) = \mathcal{G}_{a,b,c}(n_R, n_B)$ for all $n_R, n_B \in \mathbb{N}$. In fact, in the above definition we take the smallest such p and q as the row- and column-wise periods of the matrix of $\mathcal{G}$-values. The game is periodic with preperiod if the above two equations hold for sufficiently large values of n_R and n_B. We performed extensive calculations and encountered only fully periodic games. We note that there are games which exhibit other periodic behaviors leading to other definitions of periodicity, e.g., when we require only that $\mathcal{G}_{a,b,c}(n_R+p, n_B+q) = \mathcal{G}_{a,b,c}(n_R, n_B)$; cf. [2, p.58]. Note that the Grundy value sequences of one-dimensional subtraction games are periodic; cf. [2]-[3] and [7], but it is not known whether the multidimensional versions are also periodic in all cases; cf. [6].

Remark 1. We can calculate the Grundy values by the standard recurrence

$$\begin{aligned}\mathcal{G}_{a,b,c}(n_R,n_B) &= \mathcal{G}(n_R,n_B)\\ &= \operatorname{mex}\{\mathcal{G}(n'_R,n'_B) : \text{position } (n'_R,n'_B) \text{ is an option of the game } (a,b,c)\\ &\quad \text{from position } (n_R,n_B)\}\end{aligned} \tag{2.1}$$

where mex denotes the minimum excluded value operation (cf. [2]) with a slightly updated definition: with $T \subseteq \mathbb{Z}$, we set mex $T = \min\{\mathbb{N} \bigcap \neg T\}$, i.e., the smallest natural number not in T. In order to accommodate the initial values, we revise (2.1):

$$\mathcal{G}_{a,b,c}(n_R,n_B) = \mathcal{G}(n_R,n_B) = \operatorname{mex}\{\mathcal{G}(n_R-a,n_B),\mathcal{G}(n_R,n_B-b),\mathcal{G}(n_R-c,n_B+c)\}. \tag{2.2}$$

with initial values $\mathcal{G}(0,0) = 0$ and $\mathcal{G}(n_R,n_B) = -1$ if $n_R < 0$ or $n_B < 0$ or both. By the theory of Sprague–Grundy functions, the first player has a winning strategy from the initial position (n_R,n_B) exactly if $\mathcal{G}(n_R,n_B) > 0$ because these are the $\mathcal{N}$ (next player wins) positions. The winning strategy is to always move to a $\mathcal{P}$ (previous player wins) position, i.e., to a position (n'_R,n'_B) with $\mathcal{G}(n'_R,n'_B) = 0$.

Theorem 2.1. *If $a = b$ are even and $c = 1$ then in the game $(a,a,1)$ the first player wins exactly if*

$$\lfloor \frac{n_R+n_B}{a} \rfloor + n_R \equiv 1 \bmod 2. \tag{2.3}$$

In this case, all Grundy values are zeros and ones, and thus, no matter how players play the final outcome of the game is predetermined by (2.3)*. Also, for the periodicity of the Grundy values, we have*

$$\mathcal{G}(n_R+2a,n_B+2a) = \mathcal{G}(n_R,n_B+2a) = \mathcal{G}(n_R+2a,n_B) = \mathcal{G}(n_R,n_B),$$

i.e., $2a$ *rows and columns keep repeating in the matrix of Grundy numbers.*

Example 1. Table 1 shows the $\mathcal{G}$-values for the game $(2,2,1)$ with $n_R,n_B \le 10$.

Remark 2. In the original game $(2,2,1)$ we have $(n_R,n_B) = (20,14)$; thus, by (2.3), it is a win for the first player for any sequence of legal moves by the players and $(p,q) = (4,4)$. In fact, the game $(2,2,1)$ is trivial: it is a *She-loves-me, She-loves-me-not game* [2], i.e., parity is all that matters. Players don't have good and bad moves because the only $\mathcal{G}$-values are 0 and 1, all the options of 0 (if there are options) are equal to *, and all the options of * are equal to 0, in terms of the usual nimber notation; cf. [2].

Proposition 2.2. *Assume that the game $(a,b,1)$ has periods $p(a,b,1)$ and $q(a,b,1)$. With $n \in \mathbb{Z}^+$ the game (na,nb,n) is similar to the game $(a,b,1)$: first every column of the Grundy values of the game $(a,b,1)$ itself is repeated to form n identical columns, then every row of these Grundy values is repeated to form n identical rows. We have that $p(na,nb,n) = n \cdot p(a,b,1)$ and $q(na,nb,n) = n \cdot q(a,b,1)$.*

$n_R \backslash n_B$	0	1	2	3	4	5	6	7	8	9	10
0	0	0	1	1	0	0	1	1	0	0	1
1	1	0	0	1	1	0	0	1	1	0	0
2	1	1	0	0	1	1	0	0	1	1	0
3	0	1	1	0	0	1	1	0	0	1	1
4	0	0	1	1	0	0	1	1	0	0	1
5	1	0	0	1	1	0	0	1	1	0	0
6	1	1	0	0	1	1	0	0	1	1	0
7	0	1	1	0	0	1	1	0	0	1	1
8	0	0	1	1	0	0	1	1	0	0	1
9	1	0	0	1	1	0	0	1	1	0	0
10	1	1	0	0	1	1	0	0	1	1	0

Table 1: The $\mathcal{G}$-values of the game of $(2, 2, 1)$ with the periodic part highlighted

Theorem 2.1 and Proposition 2.2 imply the following

Corollary 2.3. *In case of the game $(2na, 2na, n)$ all Grundy values are zeros and ones; thus, no matter how the players play the final outcome is predetermined. The first player wins exactly if*

$$\left\lfloor \frac{\lfloor \frac{n_R}{n} \rfloor + \lfloor \frac{n_B}{n} \rfloor}{2a} \right\rfloor + \left\lfloor \frac{n_R}{n} \right\rfloor \equiv 1 \bmod 2.$$

We have the row- and column-wise periods $(p, q) = (4na, 4na)$.

For instance, for the game $(4, 4, 2)$ we have $(p, q) = (8, 8)$ and it does not matter how the two players play: the outcome depends only on $\lfloor n_R/n \rfloor$ and $\lfloor n_B/n \rfloor$.

Example 2. Table 2 shows the $\mathcal{G}$-values for the game $(4, 4, 2)$ with $n_R, n_B \le 10$.

Proposition 2.4. *In the infinite matrix*

$$(\mathcal{G}_{a,b,c}(n_R, n_B)) = (\mathcal{G}(n_R, n_B)) = \begin{pmatrix} \mathcal{G}(0,0) & \mathcal{G}(0,1) & \mathcal{G}(0,2) & \dots \\ \mathcal{G}(1,0) & \mathcal{G}(1,1) & \mathcal{G}(1,2) & \dots \\ \mathcal{G}(2,0) & \mathcal{G}(2,1) & \mathcal{G}(2,2) & \dots \\ \dots & \dots & \dots & \dots \end{pmatrix}$$

of Grundy values of the game (a, a, a) there are blocks of $a \times a$ zeros and ones alternating in the first a rows. The next a rows have blocks of $a \times a$ twos and threes alternating. The $2a$ rows and columns keep repeating.

Example 3. Table 3 shows the $\mathcal{G}$-values for the game $(2, 2, 2)$ with $n_R, n_B \le 10$.

$n_R \backslash n_B$	0	1	2	3	4	5	6	7	8	9	10
0	0	0	0	0	1	1	1	1	0	0	0
1	0	0	0	0	1	1	1	1	0	0	0
2	1	1	0	0	0	0	1	1	1	1	0
3	1	1	0	0	0	0	1	1	1	1	0
4	1	1	1	1	0	0	0	0	1	1	1
5	1	1	1	1	0	0	0	0	1	1	1
6	0	0	1	1	1	1	0	0	0	0	1
7	0	0	1	1	1	1	0	0	0	0	1
8	0	0	0	0	1	1	1	1	0	0	0
9	0	0	0	0	1	1	1	1	0	0	0
10	1	1	0	0	0	0	1	1	1	1	0

Table 2: The $\mathcal{G}$-values of the game of $(4, 4, 2)$ with the periodic part highlighted

$n_R \backslash n_B$	0	1	2	3	4	5	6	7	8	9	10
0	0	0	1	1	0	0	1	1	0	0	1
1	0	0	1	1	0	0	1	1	0	0	1
2	2	2	3	3	2	2	3	3	2	2	3
3	2	2	3	3	2	2	3	3	2	2	3
4	0	0	1	1	0	0	1	1	0	0	1
5	0	0	1	1	0	0	1	1	0	0	1
6	2	2	3	3	2	2	3	3	2	2	3
7	2	2	3	3	2	2	3	3	2	2	3
8	0	0	1	1	0	0	1	1	0	0	1
9	0	0	1	1	0	0	1	1	0	0	1
10	2	2	3	3	2	2	3	3	2	2	3

Table 3: The $\mathcal{G}$-values of the game of $(2, 2, 2)$ with the periodic part highlighted

By Theorem 2.1, Propositions 2.2, and 2.4 we obtain the periods explicitly for the games $(2a, 2a, 1)$ and $(2na, 2na, n)$: $(p(2a, 2a, 1), q(2a, 2a, 1)) = (4a, 4a)$ and $(p(2na, 2na, n), q(2na, 2na, n)) = (4na, 4na)$, respectively, while for the game (a, a, a) we have $(p(a, a, a), q(a, a, a)) = (2a, 2a)$.

Proposition 2.5. *If $a \geq 2$ then in the rows of the infinite matrix $(\mathcal{G}_{a,1,1}(n_R, n_B))$ of Grundy values of the game $(a, 1, 1)$, single zeros and ones alternate in the first a rows followed by a single row of alternating single twos and threes. These $a + 1$ rows are repeated after the zeros and ones are switched. The $2(a + 1)$ rows and two columns keep repeating.*

Example 4. Table 4 shows the $\mathcal{G}$-values for the game $(2, 1, 1)$ with $n_R, n_B \leq 10$.

Remark 3. It is clear that $\mathcal{G}_{a,b,c}(n_R, n_B) \leq 3$ since there are no more than 3 options for legal moves. This fact can be easily proven row by row by induction on

$n_R \backslash n_B$	0	1	2	3	4	5	6	7	8	9	10
0	0	1	0	1	0	1	0	1	0	1	0
1	0	1	0	1	0	1	0	1	0	1	0
2	2	3	2	3	2	3	2	3	2	3	2
3	1	0	1	0	1	0	1	0	1	0	1
4	1	0	1	0	1	0	1	0	1	0	1
5	2	3	2	3	2	3	2	3	2	3	2
6	0	1	0	1	0	1	0	1	0	1	0
7	0	1	0	1	0	1	0	1	0	1	0
8	2	3	2	3	2	3	2	3	2	3	2
9	1	0	1	0	1	0	1	0	1	0	1
10	1	0	1	0	1	0	1	0	1	0	1

Table 4: The $\mathcal{G}$-values of the game of $(2, 1, 1)$ with the periodic part highlighted

n_R. Remark 4 shows, however, that the upper bound changes if c is allowed to be a set.

3. Periodicity

We believe that all games (a, b, c) are periodic possibly with an initial preperiod. Tables of $\mathcal{G}$-values, based on numerical experimentation, suggest the following conjecture.

Conjecture 1. If $a = b \geq 3$ are odd and $c = 1$ then in the game $(a, a, 1)$ we have that

$$\mathcal{G}(n_R + p, n_B + q) = \mathcal{G}(n_R, n_B + q) = \mathcal{G}(n_R + p, n_B) = \mathcal{G}(n_R, n_B) \tag{3.1}$$

with $p = 2a(a + 1)$ and $q = 2a$.

The game $(1, 1, 1)$ is taken care of by Proposition 2.4 and has periods $p = q = 2$.

Conjecture 2. If $b \geq 2$ then in the game $(1, b, 1)$ we have that $(p, q) = (4b, 2b)$.

Example 5. Table 5 shows the $\mathcal{G}$-values for the game $(1, 2, 1)$ with $n_R, n_B \leq 10$.

Conjecture 3. If $a = b \geq 2$ and we allow that the third parameter c be a set $c = \{1, 2, \ldots, a\}$ then we have for $\mathcal{G}(n_R, n_B) = \mathcal{G}_{a,a,\{1,2,\ldots,a\}}(n_R, n_B)$ that $p = a+2$ and $q = 2a$ in (3.1).

Remark 4. If c is allowed to be a set then we have $\mathcal{G}_{a,b,c}(n_R, n_B) \leq |c|+2$.

$n_R \backslash n_B$	0	1	2	3	4	5	6	7	8	9	10
0	0	0	1	1	0	0	1	1	0	0	1
1	1	2	0	3	1	2	0	3	1	2	0
2	0	1	1	0	0	1	1	0	0	1	1
3	2	0	3	1	2	0	3	1	2	0	3
4	1	1	0	0	1	1	0	0	1	1	0
5	0	2	1	3	0	2	1	3	0	2	1
6	1	0	0	1	1	0	0	1	1	0	0
7	2	1	3	0	2	1	3	0	2	1	3
8	0	0	1	1	0	0	1	1	0	0	1
9	1	2	0	3	1	2	0	3	1	2	0
10	0	1	1	0	0	1	1	0	0	1	1

Table 5: The $\mathcal{G}$-values of the game of $(1, 2, 1)$ with the periodic part highlighted

4. Proofs

Proof of Theorem 2.1. In the game $(a, a, 1)$ with an even a the entries $(\mathcal{G}(0, n_B))_{n_B=0}^{\infty}$ in row 0 of the matrix of $\mathcal{G}$-values form a block of a zeros alternating with a block of a ones. Row 1 starts with $a-1$ ones followed by a block of a zeros alternating with a block of a zeros. Row 2 starts with $a-2$ zeros followed by a block of a ones alternating with a block of a zeros, etc. Row a has a block of a ones alternating with a block of a zeros. Row $a+1$ starts with $a-1$ zeros followed by a block of a ones alternating with a block of a zeros. Row $a+2$ starts with $a-2$ ones followed by a block of a zeros alternating with a block of a ones, etc. This pattern continues in $2a$ rows with the above $2a$ rows, i.e, rows $0, 1, \ldots, 2a-1$, periodically repeated.

It is easy to see that (2.3) is satisfied for the block of values (n_R, n_B) with $0 \leq n_R \leq 2a-1$ and $0 \leq n_B \leq 2a-1$ and then the periodicity guarantees that (2.3) holds for all n_R and n_B. $\square$

Proof of Proposition 2.2. The only difference between the Grundy values of the games (na, nb, n) and $(a, b, 1)$ is that the rows and columns of the latter one are copied immediately following the particular rows and columns to form blocks of n identical rows and columns. See Example 2 for illustration. $\square$

Proof of Proposition 2.4. Since $c = a$ we have identical rows $0, 1, \ldots, a-1$ in which blocks of a zeros and ones alternate. In a similar fashion, in the next a rows blocks of a twos and threes alternate. Since $\text{mex}\{2, 3\} = 0$ and $\text{mex}\{0, 2, 3\} = 1$, the above pattern of $2a$ rows keep repeating. $\square$

Proof of Proposition 2.5. The proof is very similar to that of Proposition 2.4 with $a = 1$ except that row 0 is repeated in order to form a identical rows of simply alternating zeros and ones. Having appended a row of alternating twos and threes, the first $a+1$ rows are repeated with the zeros and ones interchanged. Finally, the $2(a+1)$ rows keep repeating. $\square$

Acknowledgment. The author wishes to thank Gregory P. Tollisen and the referee for their helpful comments and suggestions that improved the presentation of the paper.

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