On the asymptotic analysis of a class of linear recurrences

Thomas Prellberg

thomas.prellberg@tu-clausthal.de

TU Clausthal

Problems in Combinatorial Enumeration

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- Number of partitions of a set into subsets
  - Bell Numbers

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  - Bell Numbers
- Partition lattice chains (Babai, Lengyel)
  - Lengyel's Constant
- Analysis of a recursive Program (Knuth)
  - $t(x, y, z) =$ **if** $x \leq y$ **then** y **else**
    $t(t(x-1, y, z), t(y-1, z, x), t(z-1, x, y))$

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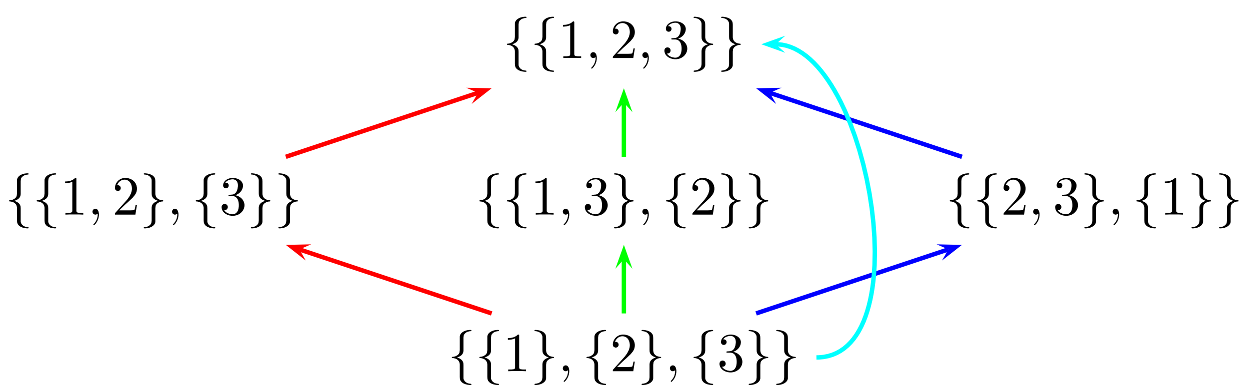
- Scale: $w \exp(w) = n$ Lambert W-function

Partition Lattice Chains

- Poset of partitions of an n-set



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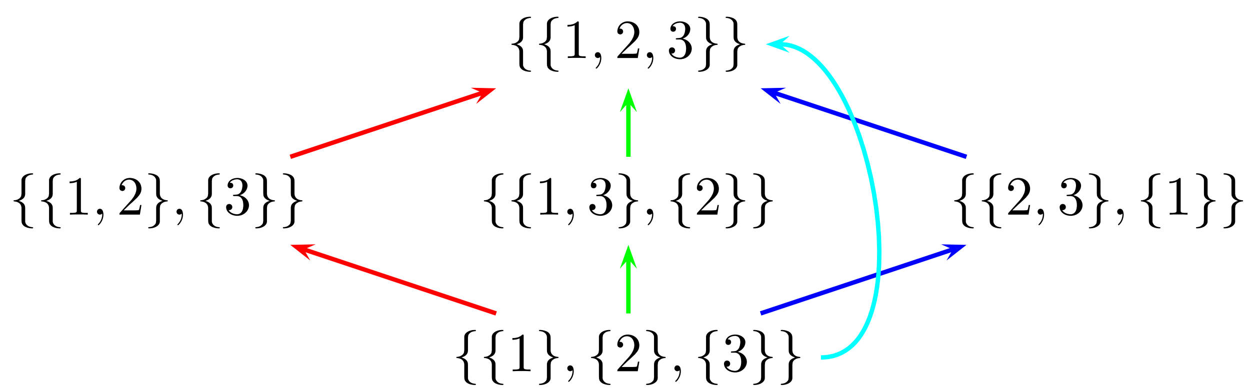

- Z_n number of chains from minimal to maximal element

$$Z_1 = 1, \quad Z_2 = 1, \quad , Z_3 = 4, \quad Z_4 = 32, \quad \ldots$$

Partition Lattice Chains (ctd.)

- Recurrence (Lengyel):

$$Z_n = \sum_{k=1}^{n-1} S_{n,k} Z_k \ , \quad S_{n,k} \text{ Stirling numbers 2nd kind}$$

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- Lengyel's Constant (Flajolet, Salvy): $C_{\mathrm{Lengyel}} = 1.0986858055\ldots$

Takeuchi Numbers

- Recursive function (Takeuchi):

$$t(x,y,z) = \textbf{if } x \le y \textbf{ then } y \textbf{ else}$$
$$t(t(x-1,y,z), t(y-1,z,x), t(z-1,x,y))$$

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- Actual value of $t(x,y,z)$ is irrelevant

$$t(x,y,z) = \begin{cases} & y & x \le y \\ \Bigg\{ & z & y \le z \\ & x & \text{else} \end{cases}$$

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- Recurrence (Knuth):

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- Functional equation for OGF (Knuth):

$$T(z) = zC(z)\,T(zC(z)) + \frac{C(z)-1}{1-z}\,, \quad C(z) = \sum_{k=0}^{\infty} \binom{2k}{k} \frac{z^k}{k+1}$$

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- Asymptotic growth (Prellberg):

$$T_n \sim C_{\text{Takeuchi}} B_n \exp \frac{1}{2} W(n)^2\,, \quad C_{\text{Takeuchi}} = 2.2394331040\ldots$$

Common Features

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- Caveat: divergence of GF!

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- Saddle point analysis

Formal Power Series Solution

Let the FPS $X(z)$ satisfy

$$X(z) = a(z)X \circ f(z) + b(z)$$

with $a(z)$, $f(z)$, and $b(z)$ analytic near $z = 0$ and

$$f(z) = z + cz^2 + dz^3 + \dots \,, \quad c > 0$$

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Then

$$X(z) = \sum_{m=0}^{\infty} \left(\prod_{k=0}^{m-1} a \circ f^k(z) \right) b \circ f^m(z)$$

Inversion via Cauchy Formula

From

$$X(z) = \sum_{m=0}^{\infty} \left(\prod_{k=0}^{m-1} a \circ f^k(z) \right) b \circ f^m(z)$$

we compute

$$X_n = [z^n]X(z) = \sum_{m=0}^{\infty} X_{n,m}$$

with

$$X_{n,m} = \frac{1}{2\pi i} \oint \left(\prod_{k=0}^{m-1} a \circ f^k(z) \right) b \circ f^m(z) \frac{dz}{z^{n+1}}$$

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- Needed: existence of $Y(z)$ and analyticity properties
  - Analytic iteration theory (Milnor, Beardon)

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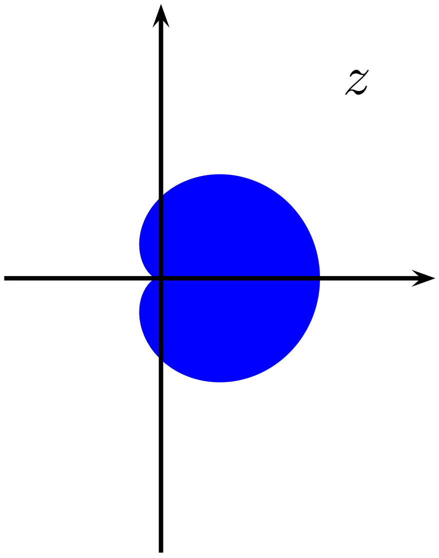

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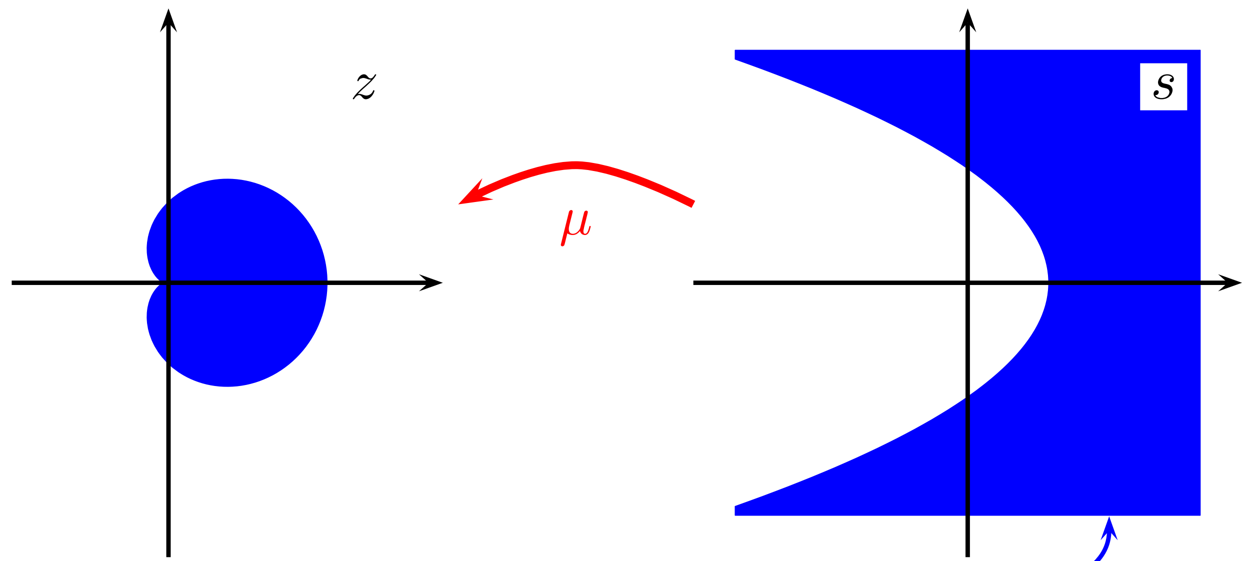

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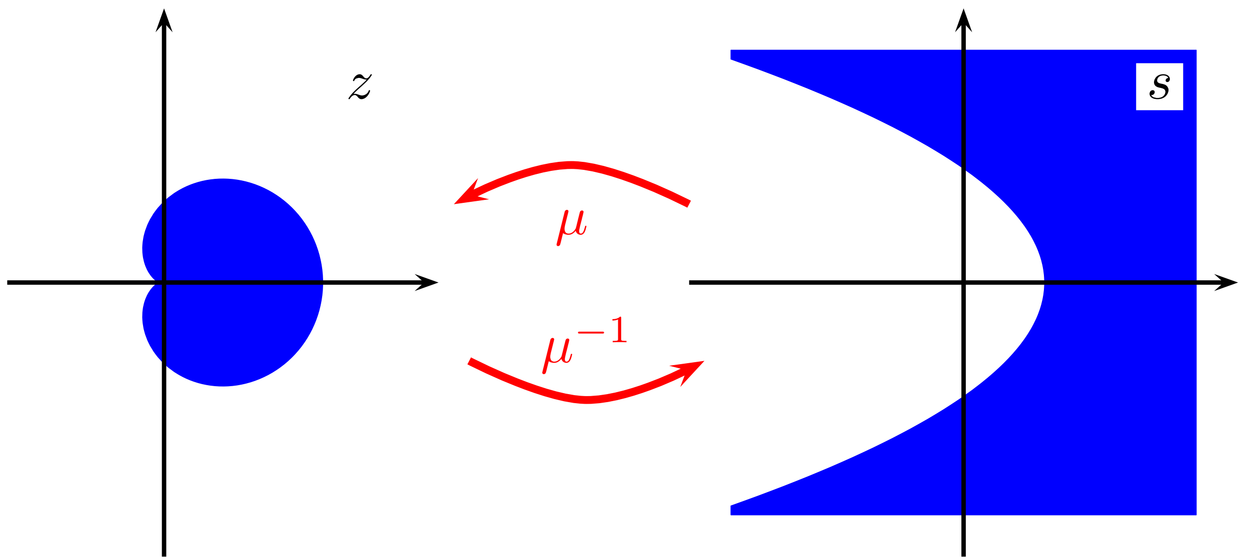

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- $f^k(z) = \mu\left(\mu^{-1}(z) - k\right)$ for z sufficiently small

Analytic Iteration Theory (ctd.)

- $\mu(s)$ admits a complete asymptotic expansion for $\Re(s) \to \infty$:

$$\mu(s) \sim \frac{1}{cs}\left(1 + \left(1 - \frac{d}{c^2}\right)\frac{\log s}{s} + \sum_{k=2}^{\infty}\sum_{j=0}^{k}\mu_{j,k}\frac{(\log s)^j}{s^k}\right)$$

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- $f^{-m} \circ \mu(s) = \mu(s+m)$ admits a complete asymptotic expansion for $m \to \infty$:

$$\mu(s+m) \sim \frac{1}{cm}\left(1 + \left(1 - \frac{d}{c^2} - s\right)\frac{\log m}{m} + \sum_{k=2}^{\infty}\sum_{j=0}^{k}\nu_{j,k}(s)\frac{(\log m)^j}{m^k}\right)$$

Solution of the Homogeneous Eqn.

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$$Y(z) = a(z)Y \circ f(z) \quad \Longrightarrow \quad Y \circ \mu(s) = a \circ \mu(s) Y \circ \mu(s-1)$$

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$\Gamma(s) = (s-1)\Gamma(s-1)$ with solution $\Gamma(s) = \lim_{n\to\infty} \frac{1 \cdot 2 \cdot \ldots \cdot n \cdot n^s}{s(s+1)\ldots(s+n)}$

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By analogy

$$Y \circ \mu(s) = \lim_{n \to \infty} \frac{a \circ \mu(1) a \circ \mu(2) \ldots a \circ \mu(n) \left(a \circ \mu(n)\right)^s}{a \circ \mu(s+1) a \circ \mu(s+2) \ldots a \circ \mu(s+n)}$$

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in analogy with $\frac{\Gamma(s+n)}{\Gamma(n)} \sim n^s$ one gets $\frac{Y \circ \mu(s+n)}{Y \circ \mu(n)} \sim (a \circ \mu(n))^s$

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- Asymptotics as $n \to \infty$:

$$\frac{Y \circ \mu(s+n)}{Y \circ \mu(n)} \sim \left(a \circ \mu(n)\right)^s$$

Asymptotics of $X_{n,m}$

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- Asymptotics of $Y \circ \mu(s+m) \sim Y \circ \mu(m)(a \circ \mu(m))^s$:

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- Asymptotics of $Y \circ \mu(s+m)$:

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- Asymptotics of $\mu(s+m) \sim \frac{1}{cm} \left(1 + \left(1 - \frac{d}{c^2} - s\right) \frac{\log m}{m} + \dots\right)$:

$$\sim \frac{Y \circ \mu(m)}{2\pi i} \int_{\mathcal{C}} \frac{b \circ \mu(s)}{Y \circ \mu(s)} (a \circ \mu(m))^s (cm)^n m^{-1-(1-\frac{d}{c^2})\frac{n}{m}} e^{\frac{n}{m}s} ds$$

Asymptotics of $X_{n,m}$

$$X_{n,m} = \frac{1}{2\pi i}\oint \frac{b\circ f^m(z)}{Y\circ f^m(z)} Y(z)\frac{dz}{z^{n+1}}$$

- Substitute $z = \mu(s+m)$:

$$\sim \frac{1}{2\pi i}\int_{\mathcal{C}} \frac{b\circ\mu(s)}{Y\circ\mu(s)} Y\circ\mu(s+m)\frac{d\mu(s+m)}{\mu(s+m)^{n+1}}$$

- Asymptotics of $Y\circ\mu(s+m)$:

$$\sim \frac{Y\circ\mu(m)}{2\pi i}\int_{\mathcal{C}} \frac{b\circ\mu(s)}{Y\circ\mu(s)} (a\circ\mu(m))^s\frac{d\mu(s+m)}{\mu(s+m)^{n+1}}$$

- Asymptotics of $\mu(s+m)$:

$$\sim (cm)^n m^{-1-(1-\frac{d}{c^2})\frac{n}{m}}\frac{Y\circ\mu(m)}{2\pi i}\int_{\mathcal{C}} \frac{b\circ\mu(s)}{Y\circ\mu(s)} \left(a\circ\mu(m)e^{\frac{n}{m}}\right)^s ds$$

Asymptotics of X_n

$$X_n = \sum_{m=0}^{\infty} X_{n,m} \, , \quad X_{n,m} \sim \ldots \int_{\mathcal{C}} \frac{b \circ \mu(s)}{Y \circ \mu(s)} \left(a \circ \mu(m) e^{\frac{n}{m}} \right)^s ds$$

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Saddle at $a \circ \mu(m) e^{\frac{n}{m}} = 1$

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- Sum simplifies to

$$X_n \sim C \sum_m (cm)^n \frac{Y \circ \mu(m)}{m} \left(a \circ \mu(m)\right)^{(1-\frac{d}{c^2}) \log m}$$

with

$$C = \frac{1}{2\pi i} \int_{\mathcal{C}} \frac{b \circ \mu(s)}{Y \circ \mu(s)} ds$$

The Saddle Point Condition

- $a(z) = a_k z^k + \ldots\,,\ \mu(s) \sim (cs)^{-1}$
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$$m = \frac{n}{kW(cn/ka_k^{1/k})} \qquad a_k > 0$$

Asymptotics of $Y \circ \mu(m)$

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$$Y \circ \mu(m) = a \circ \mu(m) Y \circ \mu(m-1)$$

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  - $k \geq 1$:

$$Y \circ \mu(m) \sim C_k \frac{a_k^m}{(c^m m!)^k} e^{k(1-\frac{d}{c^2})\frac{1}{2}(\log m)^2}$$

Main Results

THEOREM 1: Let the FPS $X(z) = \sum_{n=0}^{\infty} X_n z^n$ satisfy

$$X(z) = a(z) X \circ f(z) + b(z)$$

with $f(z) = z + cz^2 + dz^3 + \dots$, $a(z) = a_0 + \dots$, and $b(z)$ analytic near zero.

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If $c > 0$ and $0 < a_0 < 1$ then

$$X_n \sim D n! \, (-c/\log a_0)^n \, n^{(1-\frac{d}{c^2}) \log a_0 - 1}$$

as $n \to \infty$, where

$$D = \frac{1}{2\pi i} \int_{\mathcal{C}} \frac{b \circ \mu(s)}{Y \circ \mu(s)} ds (-\log a_0)^{-(1-\frac{d}{c^2}) \log a_0}$$

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If $c>0$ and $a_1>0$ then

$$X_n \sim Dc^n e^{-\frac{1}{2}(1-\frac{d}{c^2})W(\frac{c}{a_1}n)^2} \sum_{m=0}^{\infty} \frac{m^n}{m!}\left(\frac{a_1}{c}\right)^m$$

as $n\to\infty$, where

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Application: Bell Numbers

- Functional equation for OGF:

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insert $c = 1$, $d = 1$, $a_1 = 1$ into

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- Asymptotics:

$$B_n \sim D \sum_{m=0}^{\infty} \frac{m^n}{m!}$$

as $n \to \infty$, where

$$D = \frac{1}{2\pi i} \int_{\mathcal{C}} \Gamma(s) ds = \frac{1}{e} \quad \text{(sum of residues)}$$

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insert $c = \frac{1}{2}$, $d = \frac{1}{6}$, $a_0 = \frac{1}{2}$ into

$$\frac{Z_n}{n!} \sim Dn! \, (-c/\log a_0)^n \, n^{(1-\frac{d}{c^2})\log a_0 - 1}$$

as $n \to \infty$, where

$$D = \frac{1}{2\pi i} \int_{\mathcal{C}} \frac{b \circ \mu(s)}{Y \circ \mu(s)} ds (-\log a_0)^{-(1-\frac{d}{c^2})\log a_0}$$

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- Asymptotics:

$$Z_n \sim D(n!)^2 \, (2 \log 2)^{-n} \, n^{-1-\frac{1}{3}\log 2}$$

as $n \to \infty$, where

$$D = \frac{1}{2} \, (\log 2)^{\frac{1}{3}\log 2} \, \frac{1}{2\pi i} \int_{\mathcal{C}} 2^s \mu(s) ds = 1.0986858055\ldots$$

Application: Takeuchi Numbers

- Functional equation for OGF:

$$T(z) = zC(z)\,T(zC(z)) + \frac{C(z)-1}{1-z}\,, \quad C(z) = \sum_{k=0}^{\infty} \binom{2k}{k} \frac{z^k}{k+1}$$

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insert $c = 1$, $d = 2$, $a_1 = 1$ into

$$T_n \sim Dc^n e^{-\frac{1}{2}(1-\frac{d}{c^2})W(\frac{c}{a_1}n)^2} \sum_{m=0}^{\infty} \frac{m^n}{m!} \left(\frac{a_1}{c}\right)^m$$

with

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$$T_n \sim D \sum_{m=0}^{\infty} \frac{m^n}{m!} e^{\frac{1}{2}W(n)^2} = D' B_n e^{\frac{1}{2}W(n)^2}$$

as $n \to \infty$, where

$$D' = \frac{e}{2\pi i} \int_{\mathcal{C}} \frac{b \circ \mu(s)}{Y \circ \mu(s)} ds = 2.2394331040\ldots$$

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To be done:

- Computation of the contour integrals determining the constants