Lengyel's Constant

Set Partitions and Bell Numbers

Let S be a set with n elements. The set of all subsets of S has 2^n elements. By a **partition** of S we mean a disjoint set of nonempty subsets (called **blocks**) whose union is S. The set of all partitions of S has B_n elements, where B_n is the nth **Bell number**

$$B_n = \sum_{k=1}^{n} S_{n,k} = \frac{1}{e} \cdot \sum_{j=0}^{\infty} \frac{j^n}{j!} = \frac{d^n}{dx^n} \exp\left(e^x - 1\right)\Big|_{x=0}$$

and where $S_{n,k}$ denotes the **Stirling number of the second kind**. For example,

$$B_4 = 15, \quad S_{4,1} = 1, \quad S_{4,2} = 7, \quad S_{4,3} = 6 \quad \text{and} \quad S_{4,4} = 1.$$

We have recurrences ([1,2,3])

$$S_{n,k} = S_{n-1,k-1} + k \cdot S_{n-1,k}, \qquad S_{0,0} = 1, \quad S_{m,0} = 0 = S_{0,m} \quad \text{for } m \neq 0$$

$$B_n = \sum_{k=0}^{n-1} \binom{n-1}{k} \cdot B_k, \qquad B_0 = 1$$

and asymptotics

$$B_n \sim \frac{1}{\sqrt{n}} \cdot \lambda_n^{n+\frac{1}{2}} \cdot \exp\left(\lambda_n - n - 1\right)$$

where λ_n is defined by $\lambda_n \cdot \ln\left(\lambda_n\right) = n$. (See *Postscript* for more about this.)

Chains in the Subset Lattice of S

If U and V are subsets of S, write U<V if U is a proper subset of V. This endows the set of all subsets of S with a **partial ordering**; in fact, it is a **lattice** with maximum element S and minimum element $\varnothing$. The number of chains $\varnothing = U_0 < U_1 < U_2 < \ldots < U_{k-1} < U_k = S$ of length k is $k! \cdot S_{n,k}$. Hence the number of all chains from $\varnothing$ to S is ([1,3,4])

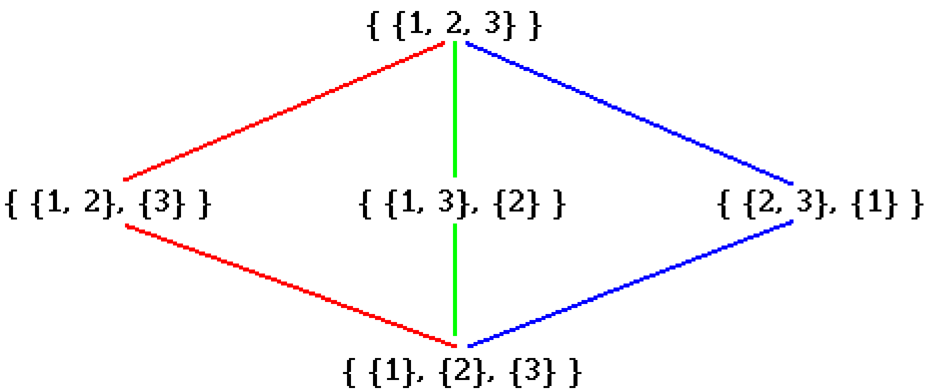
$$\sum_{k=0}^{n} k! \cdot S_{n,k} = \sum_{j=0}^{\infty} \frac{j^n}{2^{j+1}} = \frac{d^n}{dx^n} \frac{1}{2 - e^x}\Big|_{x=0} \sim \frac{n!}{2} \cdot \left(\frac{1}{\ln(2)}\right)^{n+1}$$

This is the same as the number of **ordered partitions** of S; Wilf[3] marveled at how accurate the above asymptotic approximation is. We have high accuracy here for the same reason as the fast convergence of Backhouse's constant: the generating function is meromorphic.

If one further insists that the chains are **maximal**, i.e., that each U_j has exactly j elements, then the number of such chains is n! A general technique due to P. Doubilet, G.-C. Rota and R. Stanley, involving what are called **incidence algebras**, was used in [5] to obtain the above two results (for illustration's sake). Chains can be enumerated within more complicated posets as well. As an aside, we give a deeper application of incidence algebras: to enumerating chains of linear subspaces within finite vector spaces.

Chains in the Partition Lattice of S

Nothing more needs to be said about chains in the poset of subsets of the set S. There is, however, another poset associated naturally with S which is less familar and much more difficult to study: the poset of *partitions* of S. We need first to define the partial ordering: if P and Q are two partitions of S, then P<Q if $P \neq Q$ and if $p \in P$ implies that p is a subset of q for some $q \in Q$. In other words, P is a *refinement* of Q in the sense that each of its blocks fit within a block of Q. Here is a picture for the case n=3:



For arbitrary n, the poset is, in fact, a lattice with minimum element $m = \{\{1\},\{2\},\ldots,\{n\}\}$ and maximum element $M = \{\{1, 2,\ldots,n\}\}$.

What is the number of chains $m = P_0 < P_1 < P_2 < \ldots < P_{k-1} < P_k = M$ of length k in the partition lattice of S? In the case n=3, there is only one chain for k=1, specifically, m<M. For k=2, there are three such chains and they correspond to the three distinct colors in the above picture.

Let Z_n denote the number of all chains from m to M of any length; clearly $Z_1 = Z_2 = 1$ and, by the above, $Z_3 = 4$. We have the recurrence

$$Z_n = \sum_{k=1}^{n-1} S_{n,k} \cdot Z_k$$

but techniques of Doubilet, Rota, Stanley and Bender do not apply here to give asymptotic estimates of Z_n. According to T. Lengyel[6], the partition lattice is the first natural lattice without the structure of a **binomial lattice**, which evidently implies that well-known generating function techniques are no longer helpful.

Lengyel[6] formulated a different approach to prove that the quotient

$$r(n) = \frac{Z_n}{(n!)^2 \cdot (2 \cdot \ln(2))^{-n} \cdot n^{-1-\ln(2)/3}}$$

must be bounded between two positive constants as n approaches infinity. He presented numerical evidence suggesting that r(n) tends to a unique value. Babai and Lengyel[7] then proved a fairly general convergence criterion which enabled them to conclude that

$$\Lambda = \lim_{n \to \infty} r(n) \quad \text{exists} \qquad \text{and} \qquad \Lambda = 1.09....$$

The analysis in [6] involves intricate estimates of the Stirling numbers; in [7], the focus is on nearly convex linear recurrences with finite retardation and active predecessors. Note that the *subset* lattice chains give rise to a comparatively simple asymptotic expression; *partition* lattice chains are more complicated, enough so Lengyel's constant Λ is unrecognizable and might be independent of other classical constants.

By contrast, the number of *maximal* chains is given exactly by

$$\frac{n! \cdot (n-1)!}{2^{n-1}}$$

and Lengyel[6] observed that Z_n exceeds this by an exponentially large factor.

Random Chains

Van Cutsem and Ycart[8] examined random chains in both the subset and partition lattices. It's remarkable that a common framework exists for studying these and that, in a certain sense, the limiting distributions of both types of chains are *identical*. For the sake of definiteness, let's look only at the partition lattice. If $m = P_0 < P_1 < P_2 < \ldots < P_{k-1} < P_k = M$ is the chain under consideration, let X_i denote the number of blocks in P_i (thus $X_0 = n$ and $X_k = 1$). The sequence $X_0, X_1, X_2, \ldots, X_k$ is a Markov process with known transition matrix $\Pi = \left(\pi_{i,j}\right)$ and transition probabilities

$$\pi_{i,j} = \frac{S_{i,j} \cdot Z_j}{Z_i} \qquad\qquad 1 \le i \le n\,,\ 1 \le j \le n-1$$

Note that the absorption time of this process is the same as the length k of the random chain. Among the consequences: if $\kappa_n = k/n$ is the normalized random length, then

$$\lim_{n \to \infty} E(\kappa_n) = \frac{1}{2 \cdot \ln(2)} = 0.7213475204...$$

and

$$\lim_{n \to \infty} \frac{\sqrt{n} \cdot \left(\kappa_n - \frac{1}{2 \cdot \ln(2)}\right)}{\frac{1}{2 \cdot \ln(2)} \cdot \sqrt{1 - \ln(2)}} \sim \text{Normal}(0, 1),$$

a kind of central limit theorem. Also,

$$\lim_{m \to \infty} \left(X_m - X_{m+1} - 1\right) \sim \text{Poisson}(\ln(2))$$

and hence the number of blocks in successive levels of the chain decrease slowly: the difference is 1 in 69.3% of the cases, 2 in 24.0% of the cases, 3 or more in 6.7% of the cases.

Closing Words

P. Flajolet and B. Salvy[9] have computed $\Lambda = 1.0986858055...$ to eighteen digits. Their approach is based on (fractional) analytic iterates of $\exp(x) - 1$, the functional equation

$$\phi(x) = \frac{x}{2} + \frac{1}{2} \cdot \phi\left(e^x - 1\right),$$

asymptotic expansions, the complex Laplace-Fourier transform and more. The paper is unfortunately not yet completed and a detailed report will have to wait. S. Plouffe gives all known decimal digits in the Inverse Symbolic Calculator pages.

The Mathcad PLUS 6.0 file sbsts.mcd verifies the recurrence and asymptotic results given above, and demonstrates how slow the convergence to Lengyel's constant Λ is. For more about enumerating subspaces and chains of subspaces in the vector space $F_{q,n}$, look at the 6.0 file sbspcs.mcd. (Click here if you have 6.0 and don't know how to view web-based Mathcad files).

Postscript

De Bruijn[14] gives two derivations of a more explicit asymptotic formula for the Bell numbers:

$$\frac{\ln(B_n)}{n} = \ln(n) - \ln(\ln(n)) - 1 + \frac{\ln(\ln(n))}{\ln(n)} + \frac{1}{\ln(n)} + \frac{1}{2} \cdot \left(\frac{\ln(\ln(n))}{\ln(n)}\right)^2 + O\left(\frac{\ln(\ln(n))}{\ln(n)^2}\right),$$

one by Laplace's method and the other by the saddle point method.

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S. Finch