

# ON THE DIVISIBILITY BY 2 OF THE STIRLING NUMBERS OF THE SECOND KIND

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## 1. INTRODUCTION

In this paper we characterize the divisibility by 2 of the Stirling numbers of the second kind,  $S(n, k)$ , where  $n$  is a sufficiently high power of 2. Let  $v_2(r)$  denote the highest power of 2 that divides  $r$ . We show that there exists a function  $L(k)$  such that, for all  $n \geq L(k)$ ,  $v_2(k!S(2^n, k)) = k - 1$  hold, independently from  $n$ . (The independence follows from the periodicity of the Stirling numbers *modulo* any prime power.) For  $k \geq 5$ , the function  $L(k)$  can be chosen so that  $L(k) \leq k - 2$ . We determine  $v_2(k!S(2^n + u, k))$  for  $k > u \geq 1$ , in particular for  $u = 1, 2, 3$ , and 4. We show how to calculate it for negative values, in particular for  $u = -1$ . The characterization is generalized for  $v_2(k!S(c \cdot 2^n + u, k))$ , where  $c > 0$  denotes an arbitrary odd integer.

## 2. PRELIMINARIES

The Stirling number of the second kind  $S(n, k)$  is the number of partitions of  $n$  distinct elements into  $k$  nonempty subsets. The classical divisibility properties of the Stirling numbers are usually proved by combinatorial and number theoretical arguments. Here, we combine these approaches. Inductive proofs [1] and the generating function method [10] and [7] can also be used to prove congruences among combinatorial numbers. We note that Clarke [2] used an application of  $p$ -adic integers to obtain results on the divisibility of Stirling numbers.

We define the integer-valued *order* function,  $v_a(r)$ , for all positive integers  $r$  and  $a > 1$  by  $v_a(r) = q$ , where  $a^q | r$  and  $a^{q+1} \nmid r$ , i.e.,  $v_a(r)$  denotes the highest power of  $a$  that divides  $r$ . In this paper we are interested in characterizing  $v_a(r)$ , where  $r = k!S(n, k)$  and  $a = 2$ . In a future paper, we plan to give a lower bound on  $v_a(k!S(n, k))$  for  $a \geq 3$ .

Lundell [10] discussed the divisibility by powers of a prime of the greatest common divisor of the set  $\{k!S(n, k), m \leq k \leq n\}$  for  $1 \leq m \leq n$ . Other divisibility properties have been found by Nijenhuis & Wilf [11], and recently these results have been improved by Howard [5]. Davis [3] gives a method to determine the highest power of 2 that divides  $S(n, 5)$ , i.e.,  $v_2(S(n, 5))$ . A similar method can be applied for  $S(n, 6)$  according to Davis.

We will use the well-known recurrence relation for  $S(n, k)$ , which can be proved by the inclusion-exclusion principle

$$k!S(n, k) = \sum_{i=0}^k (-1)^{k-i} \binom{k}{i} i^n. \quad (1)$$

For each prime number  $p$  and  $1 \leq i \leq p-1$ ,  $i^p \equiv i \pmod{p}$ , by Fermat's theorem, and this implies [1] that, for  $2 \leq k \leq p-1$ ,  $S(p, k) \equiv 0 \pmod{p}$ . We note that  $S(p, 1) = S(p, p) = 1$ .

Let  $d(k)$  be the sum of the digits in the binary representation of  $k$ . Using a lemma by Legendre [9], we get  $v_2(k!) = k - d(k)$ .

Note that, for  $1 \leq k \leq 4$ , identity (1) implies that  $v_2(S(2^n, k)) = d(k) - 1$ . By other identities for Stirling numbers (cf. Comtet [1], p. 227),  $v_2(S(2^n, k)) = d(k) - 1$  for  $k, 2^n - 3 \leq k \leq 2^n$ .

Classical combinatorial quantities (factorials, Bell numbers, Fibonacci numbers, etc.) often form sequences that eventually become *periodic* modulo any integer, as pointed out by I. Gessel. The "vertical" sequence of the Stirling numbers of the second kind,  $\{S(n, k) \pmod{p^N}\}_{n \geq 0}$  is periodic, i.e., there exist  $n_0 \geq k$  and  $\pi \geq 1$  such that  $S(n + \pi, k) \equiv S(n, k) \pmod{p^N}$  for  $n \geq n_0$ .

For  $N = 1$ , the minimum period was given by Nijenhuis & Wilf [11], and this result was extended for  $N > 1$  by Kwong ([7], Theorems 3.5-3.6). From now on,  $\pi(k; p^N)$  denotes the minimum period of the sequence of Stirling numbers  $\{S(n, k)\}_{n \geq k}$  modulo  $p^N$ , and  $n_0(k, p^N) \geq k$  stands for the smallest number of nonrepeating terms. Clearly,  $n_0(k, p^N) \leq n_0(k, p^{N+1})$ . Kwong proved

**Theorem A (Kwong [7]):** For  $k > \max\{4, p\}$ ,  $\pi(k; p^N) = (p-1)p^{N+b(k)-2}$ , where  $p^{b(k)-1} < k \leq p^{b(k)}$ , i.e.,  $b(k) = \lceil \log_p k \rceil$ .

From now on, we assume that  $p = 2$ ,  $n \geq 1$  and apply Theorem A for this case. Let  $g(k) = d(k) + b(k) - 2$  and  $c$  denote an odd integer. Identity (1) implies  $v_2(S(c \cdot 2^n, k)) = d(k) - 1$  for  $1 \leq k \leq \min\{4, c \cdot 2^n\}$ . We also set  $f(k) = f_c(k) = \max\{g(k), \lceil \log_2(n_0(k, 2^{d(k)})/c) \rceil\}$ . Therefore,  $c \cdot 2^{f(k)} \geq n_0(k, 2^{d(k)})$ . We note that  $g(k) \leq 2\lceil \log_2 k \rceil - 2$ . Lemma 3 in [8] yields  $f(2^m) = m$  for  $m \geq 1$  and  $c = 1$ .

In this paper we prove

**Theorem 1:** For all positive integers  $k$  and  $n$  such that  $n \geq f(k)$ , we have  $v_2(k!S(c \cdot 2^n, k)) = k - 1$  or, equivalently,  $v_2(S(c \cdot 2^n, k)) = d(k) - 1$ .

Numerical evidence suggests that the range might be extended for all  $n$  provided  $2^n \geq k$  and  $c = 1$ . For example, for  $k = 7$ , we get  $g(7) = d(7) + b(7) - 2 = 4$  and  $n_0(7, 2^3) = 7$ ; therefore, by Theorem 1, if  $n \geq f(7) = 4$ , then  $v_2(S(2^n, 7)) = v_2(S(c \cdot 2^n, 7)) = 2$  for arbitrary positive integer  $c$ . Notice, however, that  $v_2(S(8, 7)) = 2$  also. We make the following

**Conjecture:** For all  $k$  and  $1 \leq k \leq 2^n$ , we have  $v_2(S(2^n, k)) = d(k) - 1$ .

By Theorem 1, the Conjecture is true for all  $k = 2^m$  with  $m \leq n$ .

In section 3 we prove Theorem 2, which gives the exact order of  $S(n, k)$  in a particular range for  $k$  whose size depends on  $v_2(n)$ . Theorem 2 is the key tool in proving Theorem 1. Its proof makes use of the periodicity of the Stirling numbers. It would be interesting to determine the function  $L(k)$ , which is defined as the smallest integer  $n'$  such that  $v_2(S(c \cdot 2^n, k)) = d(k) - 1$  for all  $n \geq n'$ . By Theorem 2, we find that  $L(k) \leq k - 2$  and Theorem 1 improves the upper bound on  $L(k)$  if  $f(k) < k - 2$ .

In section 4 we obtain some consequences of Theorem 2 by extending it for Stirling numbers of the form  $S(c \cdot 2^n + u, k)$ , where  $u = 1, 2$ , etc. We show how to calculate  $v_2(S(c \cdot 2^n - 1, k))$ . In neither case does the order of  $S(c \cdot 2^n + u, k)$  depend on  $n$  (if  $n$  is sufficiently large), in agreement with Theorem A.

3. TOOLS AND PROOFS

We choose an integer  $\ell$  such that  $\ell \leq n$ . We shall generalize identity (1) for any modulus of the form  $2^\ell$ . Observe that, for any  $i$  even,  $i^n \equiv 0 \pmod{2^\ell}$ , and for all  $i$  odd,  $(-1)^{k-i}$  will have the same sign as  $(-1)^{k-1}$ . Therefore, by identity (1),

$$k!S(n, k) \equiv (-1)^{k-1} \sum_{\substack{i=1 \\ i \text{ odd}}}^k \binom{k}{i} i^n \pmod{2^\ell}. \tag{2}$$

The expression on the right-hand side of congruence (2) is called the *partial Stirling number* [10]. We explore identity (2) with different choices of  $n$  in order to find  $v_2(S(n, k))$ .

We shall need the following

**Theorem 2:** Let  $c$  be an odd integer and let  $n$  be a nonnegative integer. If  $1 \leq k \leq n+2$ , then  $v_2(k!S(c \cdot 2^n, k)) = k-1$ , i.e.,  $v_2(S(c \cdot 2^n, k)) = d(k) - 1$ .

Roughly speaking, Theorem 2 gives the exact value of  $v_2(k!S(m, k))$ , for  $k \geq 2$ , if  $m$  is divisible by  $2^{k-2}$ . The higher the power of 2 that divides  $m$ , the larger the value of  $k$  that can be used. We prove Theorem 1 and then return to the proof of Theorem 2.

**Proof of Theorem 1:** Without loss of generality, we assume that  $k > 4$ . Observe that  $v_2(S(c \cdot 2^n, k)) = d(k) - 1$  is equivalent to

$$S(c \cdot 2^n, k) \equiv 0 \pmod{2^{d(k)-1}} \tag{3}$$

and

$$S(c \cdot 2^n, k) \not\equiv 0 \pmod{2^{d(k)}}. \tag{4}$$

The proof of identities (3) and (4) is by contradiction. To prove the former identity, we set  $N = d(k) - 1$ , hence Theorem A yields

$$\pi(k; 2^N) = 2^{d(k)+b(k)-3} \tag{5}$$

where  $d(k) + b(k) - 3 < g(k) \leq f(k)$ .

We assume, to the contrary of the claim, that  $S(c \cdot 2^{f(k)}, k) \equiv a \not\equiv 0 \pmod{2^N}$ . By Theorem A and the period given by (5), we obtain that, for every positive integer  $m \geq c$ ,  $S(m \cdot 2^{f(k)}, k) \equiv a \not\equiv 0 \pmod{2^N}$ . This is a contradiction, for one can select  $m$  so that  $m \cdot 2^{f(k)}$  becomes  $c \cdot 2^n$ , with a large exponent  $n$ , and by Theorem 2,  $S(c \cdot 2^n, k) \equiv 0 \pmod{2^N}$  should be for sufficiently large  $n$ . It follows that, in fact,  $S(c \cdot 2^{f(k)}, k) \equiv 0 \pmod{2^N}$ , and Theorem A implies  $S(c \cdot 2^n, k) \equiv 0 \pmod{2^{d(k)-1}}$  for all  $n \geq f(k)$ .

To derive identity (4), we set  $N = d(k)$ . In order to obtain a contradiction, we assume that  $S(c \cdot 2^{f(k)}, k) \equiv 0 \pmod{2^N}$ . Now, by Theorem A, we get  $\pi(k; 2^N) = 2^{d(k)+b(k)-2}$ , where  $d(k) + b(k) - 2 = g(k) \leq f(k)$ . We proceed in a manner similar to that used above by noting that the periodicity now yields  $S(m \cdot 2^{f(k)}, k) \equiv 0 \pmod{2^N}$  for every positive integer  $m \geq c$ . It would imply that, for a sufficiently large  $n$ ,  $S(c \cdot 2^n, k) \equiv 0 \pmod{2^{d(k)}}$ . However, this congruence contradicts Theorem 2. It follows that  $S(c \cdot 2^n, k) \not\equiv 0 \pmod{2^{d(k)}}$  for  $n \geq f(k)$ , and the proof is now complete.  $\square$

**Proof of Theorem 2:** We set  $m = c \cdot 2^n$  and select an  $\ell$  such that  $1 \leq \ell \leq n+1$ . By Euler's theorem,  $\phi(2^\ell) = 2^{\ell-1}$ ; therefore,  $i^m \equiv 1 \pmod{2^\ell}$  if  $i$  is odd. By simple summation, identity (2) yields

$$k! S(m, k) \equiv (-1)^{k-1} \sum_{\substack{i=1 \\ i \text{ odd}}}^k \binom{k}{i} = (-2)^{k-1} \pmod{2^\ell}; \quad (6)$$

therefore,  $v_2(k! S(m, k)) = k - 1$ , provided  $0 \leq k - 1 < \ell$ .

We have two cases if  $k = n + 2$ . If  $m$  is odd, then  $n = 0$  and  $k = 2$ . The claim is true, since  $S(m, 2) = 2^{m-1} - 1$ ; therefore,  $v_2(2! S(m, 2)) = 1$ . If  $m$  is even, then we set  $\ell = n + 2 \geq 3$ . By induction on  $\ell \geq 3$ , we can derive  $i^{2^{\ell-2}} \equiv 1 \pmod{2^\ell}$  and identity (6) is verified again.  $\square$

**Remark:** By setting  $\ell = n + 1$ , identity (6) implies the lower bound  $v_2(k! S(c \cdot 2^n, k)) \geq n + 1$ , for  $k \geq n + 2$ .

#### 4. RELATED RESULTS

We will use other special cases of identity (2). Similarly to the previous proof, we get that, for all  $u \geq 0$ ,  $n \geq \ell \geq 1$ , and  $k \leq c \cdot 2^n + u$ ,

$$k! S(c \cdot 2^n + u, k) \equiv (-1)^{k-1} \sum_{\substack{i=1 \\ i \text{ odd}}}^k \binom{k}{i} i^{c \cdot 2^n + u} \equiv (-1)^{k-1} \sum_{\substack{i=1 \\ i \text{ odd}}}^k \binom{k}{i} i^u \pmod{2^{\ell+2}}. \quad (7)$$

We set

$$h(k, u) = (-1)^{k-1} \sum_{\substack{i=1 \\ i \text{ odd}}}^k \binom{k}{i} i^u.$$

By identity  $x^u = \sum_{j=0}^u S(u, j) \binom{x}{j} j!$ , we obtain

$$h(k, u) = (-1)^{k-1} \sum_{\substack{i=1 \\ i \text{ odd}}}^k \binom{k}{i} \sum_{j=0}^u S(u, j) \binom{i}{j} j! = (-1)^{k-1} \sum_{j=1}^{\min\{u, k\}} S(u, j) j! \sum_{\substack{i=1 \\ i \text{ odd}}}^k \binom{k}{i} \binom{i}{j}.$$

We focus on the case in which  $k > u$  and derive

$$h(k, u) = (-1)^{k-1} \sum_{j=0}^u S(u, j) j! \binom{k}{j} \sum_{\substack{i=j \\ i \text{ odd}}}^k \binom{k-j}{i-j} = (-2)^{k-1} \sum_{j=0}^u \frac{S(u, j) j!}{2^j} \binom{k}{j}. \quad (8)$$

We introduce the notation  $r(k, u) = v_2(h(k, u))$ . Identity (8) implies that  $r(k, u) \geq k - u - 1$ . Note that  $|h(k, 0)| = 2^{k-1}$  and, for  $u \geq 1$ ,

$$|h(k, u)| / 2^{k-u-1} \leq \sum_{j=1}^u j^u 2^{u-j} k^j \leq u(2u)^u (k/2)^u = u(uk)^u. \quad (9)$$

By identity (7), for  $u \geq 0$  and any sufficiently large  $\ell$  and  $n \geq \ell$ , we have  $v_2(k! S(c \cdot 2^n + u, k)) = r(k, u)$ . In fact,  $n \geq \ell = r(k, u) - 1$  will suffice; for instance,  $n \geq k - 2$  will be large enough if  $u = 0$  (Theorem 2). By identity (9), we derive that  $r(k, u) \leq k - u - 1 + u \log_2 k + (u + 1) \log_2 u$ ; therefore,  $k - u - 2 + [u \log_2 k + (u + 1) \log_2 u]$  can be chosen for  $n$  if  $u > 0$ . We note that, similarly to the proof of Theorem 1, this value might be decreased.

The values of  $r(k, u)$  can be calculated by identity (8). For example, if  $k > u \geq 0$ , then

$$r(k, u) = \begin{cases} k-1, & \text{if } u = 0, \\ k-2 + v_2(k), & \text{if } u = 1, \\ k-3 + v_2(k) + v_2(k+1), & \text{if } u = 2, \\ k-4 + 2v_2(k) + v_2(k+3), & \text{if } u = 3, \\ k-5 + v_2(k) + v_2(k+1) + v_2(k^2 + 5k - 2), & \text{if } u = 4. \end{cases} \quad (10)$$

We state two special cases that can be proved basically differently; although, in the second case, only a partial proof comes out by the applied recurrence relations.

**Theorem 3:** For  $k \geq 2$  and any sufficiently large  $n$ ,  $v_2(k!S(c \cdot 2^n + 1, k)) = k - 2 + v_2(k)$ .

**Proof:** The proof follows from Theorem 2 and using the recurrence relation  $k!S(m, k) = k\{(k-1)!S(m-1, k-1) + k!S(m-1, k)\}$  with  $m = c \cdot 2^n + 1$ . Notice that, by Theorem 1,  $n \geq \max\{f(k), f(k-1)\}$  will be sufficiently large.  $\square$

**Theorem 4:** For  $k \geq 3$  and sufficiently large  $n$ ,  $v_2(k!S(c \cdot 2^n + 2, k)) = k - 3 + v_2(k) + v_2(k+1)$ .

**Proof:** By identity (10), we obtain  $v_2(k!S(c \cdot 2^n + 2, k)) = r(k, 2) = k - 3 + v_2(k) + v_2(k+1)$ . Observe that  $n \geq \max\{f(k), f(k-1), f(k-2)\}$  suffices.  $\square$

Notice that we could have used the expansion

$$k!S(c \cdot 2^n + 2, k) = k\{(k-1)!S(c \cdot 2^n + 1, k-1) + k!S(c \cdot 2^n + 1, k)\}.$$

By Theorem 3, the first term of the second factor is divisible by a power of 2 with exponent  $k - 3 + v_2(k - 1)$ , while the second term is divisible by 2 at exponent  $k - 2 + v_2(k)$ . The first factor contributes an additional exponent of  $v_2(k)$  to the power of 2. We combine the two terms and find that there is always a unique term with the lowest exponent of 2 if  $k \not\equiv 3 \pmod{4}$ . For  $k \equiv 3 \pmod{4}$ , however, this argument falls short and we are able to obtain only the lower bound  $k - 1$  on  $v_2(k!S(c \cdot 2^n + 2, k))$ .

It turns out that calculating  $v_2(k!S(c \cdot 2^n + u, k))$  for negative integers  $u$  is more difficult than for positive values. The periodicity guarantees that the order does not depend on  $n$  (for sufficiently large  $n$ ).

We extend the function  $h(k, u)$  for negative integers  $u$ . We will choose an appropriate value  $\ell \geq 1$  and then set  $n$  so that it satisfies the inequality  $c \cdot 2^n + u \geq 2^\ell$ . We use the convenient notation  $1/i$  for the unique integer solution  $x$  of the congruence  $i \cdot x \equiv 1 \pmod{2^{\ell+2}}$  if  $i$  is odd. Similarly to identity (7), we obtain

$$k!S(c \cdot 2^n + u, k) \equiv (-1)^{k-1} \sum_{\substack{i=1 \\ i \text{ odd}}}^k \binom{k}{i} \left(\frac{1}{i}\right)^{-u} \pmod{2^{\ell+2}}. \quad (11)$$

For  $u < 0$ , we set

$$h(k, u) = (-1)^{k-1} \sum_{\substack{i=1 \\ i \text{ odd}}}^k \binom{k}{i} \left(\frac{1}{i}\right)^{-u}$$

and express  $h(k, u)$  as a fraction  $\frac{p_k(u)}{q_k(u)}$  in lowest terms. Notice that  $v_2(p_k(u)) \geq k - d(k)$  holds, since  $k!$  divides both sides of (11) for any sufficiently large  $\ell$ . The order of  $v_2(S(c \cdot 2^n + u, k))$  can be determined by choosing  $\ell \geq v_2(p_k(u)) - 1$ , and the *actual order* is  $v_2(p_k(u)) - k + d(k)$ . We remark that, for  $c = 1$ , the value of  $n$  can be set to  $v_2(p_k(u))$ .

We focus on the case of  $u = -1$ . Let

$$a_k = \sum_{i=1}^k \binom{k}{i} \frac{1}{i}.$$

We get

$$a_s - a_{s-1} - \binom{s}{s} \frac{1}{s} = \sum_{i=1}^{s-1} \frac{1}{i} \left\{ \binom{s}{i} - \binom{s-1}{i} \right\} = \sum_{i=1}^{s-1} \frac{1}{s} \binom{s}{i} = \frac{2^s - 2}{s} \quad (s \geq 2).$$

By summation, it follows that

$$a_k = \sum_{i=1}^k \frac{2^i}{i} - \sum_{i=1}^k \frac{1}{i}.$$

Similarly,

$$b_k = \sum_{i=1}^k \binom{k}{i} \frac{(-1)^{i+1}}{i} = \sum_{i=1}^k \frac{1}{i}$$

(cf. Hietala & Winter [4], or Solution to Problem E3052 in *Amer. Math. Monthly* **94.2** (1987): 185). Combining these two identities, we obtain

$$h(k, -1) = \sum_{\substack{i=1 \\ i \text{ odd}}}^k \binom{k}{i} \frac{1}{i} = \frac{1}{2} \sum_{i=1}^k \frac{2^i}{i} = \frac{p_k(-1)}{q_k(-1)}. \tag{12}$$

For example, for  $k = 5$ , we get  $h(5, -1) = \frac{128}{15}$ ,  $v_2(p_5(-1)) = 7$ , and  $n \geq 7$ . E.g.,  $v_2(S(127, 5)) = v_2(S(255, 5)) = \dots = 4$ . We remark that  $v_2(S(63, 5)) = 4$  holds, too. Notice that the recurrence relation  $S(N, K) = K \cdot S(N - 1, K) + S(N - 1, K - 1)$  implies that  $v_2(S(c \cdot 2^n - 1, 2^m - 1)) = 0$  for every sufficiently large  $n$ . By the theory of  $p$ -adic numbers [6] and (12), we can derive that, for all sufficiently large  $n$ ,

$$v_2(S(c \cdot 2^n - 1, k)) = v_2\left(\frac{1}{2} \sum_{i=1}^k \frac{2^i}{i}\right) - k + d(k) = v_2\left(\frac{1}{2} \sum_{i=k+1}^{\infty} \frac{2^i}{i}\right) - k + d(k),$$

where  $v_2(a/b)$  is defined as  $v_2(a) - v_2(b)$  if  $a$  and  $b$  are integers. This fact helps us to make observations for some special cases. For instance, if  $n > m \geq 3$ , then  $v_2(S(c \cdot 2^n - 1, 2^m)) \geq 2$  holds and, therefore,  $v_2(S(c \cdot 2^n - 1, 2^m + 1)) = 1$ . Numerical evidence suggests that, for  $n > m \geq 4$ ,  $v_2(S(c \cdot 2^n - 1, 2^m)) = 2m - 2$ , although we were unable to prove this.

We can determine  $v_2(S(c \cdot 2^n - 1, k))$  for most of the odd values of  $k$  by systematically evaluating  $v_2(\sum_{i=1}^k \frac{2^i}{i})$ , and obtain

**Theorem 5:** For all sufficiently large  $n$ ,  $v_2(S(c \cdot 2^n - 1, k)) = d(k) - v_2(k + 1)$ , if  $k \geq 1$  is odd and  $k \not\equiv 5 \pmod{8}$  and  $k \not\equiv 59 \pmod{64}$  and  $k \not\equiv 121 \pmod{128}$ .

We leave the details of the proof to the reader.

We note that there is an alternative way of determining  $p_k(-1)$ . We set

$$I_{k-1} = \frac{k}{2^{k-1}} \frac{1}{2} \sum_{i=1}^k \frac{2^i}{i}.$$

One can prove that

$$I_k = \sum_{j=0}^k \frac{1}{\binom{k}{j}} \quad \text{and} \quad I_k = \frac{k+1}{2k} I_{k-1} + 1.$$

For other properties of  $I_k$ , see Comtet ([1], p. 294, Exercise 15). The latter recurrence relation simplifies the calculation of  $v_2(S(c \cdot 2^n - 1, k))$  for large values of  $k$ .

We can use (7) in a slightly different way and gain information on the structure of the sequence  $\{S(c \cdot 2^n + k, k), S(c \cdot 2^n + k + 1, k), \dots, S((c+1) \cdot 2^n + k - 1, k) \pmod{2^q}\}$  for every  $q$ ,  $1 \leq q \leq d(k) - 1$  and sufficiently large  $n$ . We observe that the sequence always starts with a one and ends with at least  $d(k) - q$  zeros. Note that, for every  $\ell$  and  $u$  such that  $k > u \geq \ell > k - d(k)$ ,

$$0 = k!S(u, k) \equiv (-1)^{k-1} \sum_{\substack{i=1 \\ i \text{ odd}}}^k \binom{k}{i} i^u \pmod{2^\ell}.$$

We set  $q = \ell - k + d(k)$ . Clearly,  $1 \leq q \leq d(k) - 1$ . By (7), we get that  $k!S(c \cdot 2^n + u, k) \equiv 0 \pmod{2^q}$  for all  $n \geq \ell - 2 \geq 1$ . This observation yields that the  $d(k) - q$  consecutive terms,

$$S(c \cdot 2^n + u, k) \pmod{2^q}, \quad u = k - d(k) + q, k - d(k) + q + 1, \dots, k - 1, \quad (13)$$

are all zeros. Similarly, we can derive  $k!S(c \cdot 2^n + k, k) \equiv k! \not\equiv 0 \pmod{2^q}$ , i.e.,  $S(c \cdot 2^n + k, k) \equiv 1 \pmod{2^q}$ . Identities (8) and (10) imply that there might be many more zeros in the sequence at and after the term  $S(c \cdot 2^n, k) \pmod{2^q}$ .

For example, if  $k = 7$  and  $\ell = 5$ , then  $S(c \cdot 2^n + u, 7) \equiv 0 \pmod{2^1}$  for  $u = 5$  and  $6$  and for all  $n \geq 3$ . Similarly to the proof of Theorem 1, it follows that identity (13) holds if  $n \geq f(k)$ . For instance, if  $k = 23$  and  $\ell = 21$ , then  $S(c \cdot 2^n + u, 23) \equiv 0 \pmod{2^2}$  for  $u = 21$  and  $22$  provided  $n \geq f(23) = 7$ .

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