IDENTITIES FOR THE GENERATING FUNCTION OF THE MULTISET $[n\Phi^m]$ FOR $m = -1, 1, 2$

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ABSTRACT

Some remarkable identities involving the power series in which the exponents are of form $[n\alpha]$ with some irrational $\alpha > 0$ have been obtained. Here we present short proofs for some related identities with $\alpha = \Phi^m, m = -1, 1, 2$.

1. INTRODUCTION


$$\sum_{n=1}^{\infty} z^{[n\alpha]},$$

$|z| < 1$, and derived identities in terms of continued fraction expansions with partial quotients that are rational functions of $z$. This power series is also referred to as the generating function for the spectrum of the multiset $[n\alpha]$.

Most prominently, they showed that for $b > 1$ integer, $S_b(\alpha) = (b-1) \sum_{n=1}^{\infty} \frac{1}{b^n\alpha}$, can be described explicitly as the infinite simple continued fraction $[l_0, l_1, \ldots]$, with $\alpha = [a_0, a_1, \ldots]$, $n$th convergents $\frac{p_n}{q_n} = [a_0, a_1, \ldots, a_n], n \geq 0$, $q_{-1} = 0$, and $t_0 = a_0 b, t_n = \frac{t_{n-1}b^n - t_{n-2}}{b^n - 1}, n \geq 1$. Clearly,

$$\frac{z}{1 - z} S_{1/\alpha}(1/\alpha) = \sum_{n=1}^{\infty} z^{[n\alpha]}$$

for any $z$ which is the reciprocal of an integer $b > 1$.

In [3], the remarkable identity

$$\frac{z F_1}{1 + \frac{z F_2}{1 + \frac{z F_3}{1 + \cdots}}} = (1 - z) \sum_{n=1}^{\infty} z^{[n\Phi]},$$

$|z| < 1$, or in an equivalent form (using a slightly unusual continued fraction form corresponding to the so called continuant polynomials)

$$[0, z^{-F_0}, z^{-F_1}, z^{-F_2}, \ldots] = \frac{1 - z}{z} \sum_{n=1}^{\infty} z^{[n\Phi]},$$

(2)
was proven. According to [2], it can be used to derive the power series version of the identity (1) for \( \alpha = \Phi \).

2. RESULTS

We will consider and prove two specific power series identities for \(|z| < 1\),

\[
\frac{1}{1-z} = \sum_{n=1}^{\infty} z^n \text{[Frac]} - \sum_{n=1}^{\infty} z^{n\Phi} \tag{3}
\]

and

\[
\frac{1+z}{1-z} = \sum_{n=1}^{\infty} z^n \text{[Frac]} + \sum_{n=1}^{\infty} z^{n\Phi^2} \tag{4}
\]

To obtain a partial proof of (3) for any \( z = 1/b, b > 1 \) integer, we can use identity (1)

\[
\frac{z}{1-z} \left( S_{1/z}(\Phi) - S_{1/z}(1/\Phi) \right) = \frac{z}{1-z} \left( \left[ \frac{1}{z}, A \right] - [0, A] \right) = \frac{1}{1-z}
\]

with \( A \) comprising the “partial quotients” \( \frac{(\frac{1}{z})^{r_{n+1}} - (\frac{1}{z})^{r_{n-1}}}{(\frac{1}{z})^{r_n} - 1} \), \( n \geq 1 \).

To get a direct proof without using identities (1) and (2), we observe that in the first sum, each term \( z^k \) comes with a coefficient 1 or 2, and \( k = \lfloor \frac{n}{\Phi} \rfloor = \lfloor \frac{n+1}{\Phi} \rfloor \) if and only if \( \lfloor (n - \lfloor \frac{n}{\Phi} \rfloor) \Phi \rfloor = k \). In fact, if \( \lfloor \frac{n}{\Phi} \rfloor = \lfloor \frac{n+1}{\Phi} \rfloor \) then

\[
\frac{n}{\Phi} \leq \left( n - \left\lfloor \frac{n}{\Phi} \right\rfloor \right) \Phi = \left( n - \left\lfloor \frac{n+1}{\Phi} \right\rfloor \right) \Phi < n\Phi + \Phi - (n + 1) = (n + 1)(\Phi - 1) = \frac{n+1}{\Phi}
\]

since \( \frac{1}{\Phi} = \Phi - 1 \), \( 0 \leq \lfloor \frac{n}{\Phi} \rfloor b \leq a \), and \( 0 \leq a - \lfloor \frac{n}{\Phi} \rfloor b < b \) with \( a, b \geq 0 \), and we can take the integer parts. In other words, there is a term \( z^k \) to be subtracted in the second sum in (3). On the other hand, if \( k = \lfloor r\Phi \rfloor \) for some integer \( r \geq 1 \) (i.e., \( k - 1 \) is Fibonacci even [3] then with \( n = \lfloor k\Phi \rfloor + 1 \) we get \( \lfloor \frac{n}{\Phi} \rfloor = \lfloor \frac{n+1}{\Phi} \rfloor = k \) to the same effect.

To prove identity (4), we observe that, as \( \frac{1}{\Phi} + \frac{1}{\Phi^2} = 1 \), Beatty’s theorem guarantees that \( a_n = \lfloor n\Phi \rfloor \) and \( b_n = \lfloor n\Phi^2 \rfloor = a_n + n \), \( n \geq 1 \), form a set in which each positive integer occurs exactly once. This yields

\[
\frac{z}{1-z} = \sum_{n=1}^{\infty} z^n \text{[Frac]} + \sum_{n=1}^{\infty} z^{n\Phi^2}, \tag{5}
\]

which implies (4) via (3).
REFERENCES


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