ON THE DIVISIBILITY BY 2 OF THE STIRLING NUMBERS OF THE SECOND KIND

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1. Introduction

In this paper we characterize the divisibility by 2 of the Stirling number of the second kind, \( S(n, k) \), where \( n \) is a sufficiently high power of 2. Let \( \nu_2(r) \) denote the highest power of 2 which divides \( r \). We show that there exists a function \( L(k) \) such that for all \( n \geq L(k) \), \( \nu_2(k!S(2^p, k)) = k - 1 \) hold, independently from \( n \). (Here the independence follows from the periodicity of the Stirling numbers modulo any prime power.) For \( k \geq 5 \), the function \( L(k) \) can be chosen so that \( L(k) \leq k - 2 \). We determine \( \nu_2(k!S(2^p + u, k)) \) for \( k > u \geq 1 \), in particular for \( u = 1, 2, 3, \) and 4. We show how to calculate it for negative values, in particular for \( u = -1 \). The characterization is generalized for \( \nu_2(k!S(c \cdot 2^p + u, k)) \) where \( c > 0 \) denotes an arbitrary odd integer.

2. Preliminaries

The Stirling number of the second kind \( S(n, k) \) is the number of partitions of \( n \) distinct elements into \( k \) non-empty subsets. The classical divisibility properties of the Stirling numbers are usually proved by combinatorial and number theoretical arguments. Here we combine these approaches. Inductive proofs [1] and the generating function method ([11] and [17]) can also be used to prove congruences among combinatorial numbers. We note that Clarke [2] used an application of \( p \)-adic integers to obtain results on the divisibility of Stirling numbers.

We define the integer-valued order function, \( \nu_a(r) \), for all positive integers \( r \) and \( a > 1 \) by \( \nu_a(r) = q \), where \( a^q \mid r \), and \( a^{q+1} \nmid r \), i.e., \( \nu_a(r) \) denotes the highest power of \( a \) which divides \( r \). In this paper we are interested in characterizing \( \nu_a(r) \), where \( r = k!S(n,k) \) and \( a = 2 \). In [10] we give a lower bound on \( \nu_a(k!S(n,k)) \) for \( a \geq 3 \).

Lundell [11] discussed the divisibility by powers of a prime of the greatest common divisor of the set \( \{k!S(n, k), m \leq k \leq n\} \), for \( 1 \leq m \leq n \). Other divisibility properties have been found by Nijenhuis and Wilf [12], and recently these results have been improved by Howard [5]. Davis [3] gives a method to determine the highest power of 2 which divides \( S(n, 5) \), i.e., \( \nu_2(S(n, 5)) \). A similar method can be applied for \( S(n, 6) \) according to Davis.

We will use the well known recurrence relation for \( S(n, k) \) which can be proved by the inclusion–exclusion principle

\[
k!S(n, k) = \sum_{i=0}^{k} (-1)^{k-i} \binom{k}{i} i^p.
\]

For each prime number \( p \) and \( 1 \leq i \leq p - 1 \), \( i^p \equiv i \pmod{p} \) by Fermat’s theorem, and this implies [1] that, for \( 2 \leq k \leq p - 1 \), \( S(p, k) \equiv 0 \pmod{p} \). We note that \( S(p, 1) = S(p, p) = 1 \).

Let \( d(k) \) be the sum of the digits in the binary representation of \( k \). Using a lemma by Legendre [9], we get \( \nu_2(k!S(n, k)) = d(k) - 1 \).

Note that, for \( 1 \leq k \leq 4 \), identity (1) implies that \( \nu_2(S(2^p, k)) = d(k) - 1 \). By other identities for Stirling numbers (cf. Comtet [1], p. 227), \( \nu_2(S(2^p, k)) = d(k) - 1 \) for \( k \), \( 2^q - 3 \leq k \leq 2^q \).

Classical combinatorial quantities (e.g., factorials, Bell numbers, Fibonacci numbers, etc.) often form sequences that eventually become periodic modulo any integer as it was pointed out by I. Gessel. The “vertical” sequence of the Stirling numbers of the second kind, \( \{S(n, k) \pmod{p^N}\}_{n \geq 0} \) is periodic, i.e., there exist \( n_0 \geq k \) and \( \pi \geq 1 \) such that \( S(n + \pi, k) \equiv S(n, k) \pmod{p^N} \) for \( n \geq n_0 \).
For $N = 1$, the minimum period was given by Nijenhuis and Wilf [12], and this result was extended for $N > 1$ by Kwong ([7], Theorems 3.5 and 3.6). From now on $\pi(k; p^N)$ denotes the minimum period of the sequence of Stirling numbers $\{S(n, k)\}_{n \geq k}$ modulo $p^N$, and $n_0(k, p^N) \geq k$ stands for the smallest number of nonrepeating terms. Clearly $n_0(k, p^N) \leq n_0(k, p^{N+1})$. Kwong proved

Theorem A. (Kwong [7]) For $k > \max\{4, p\}$, $\pi(k; p^N) = (p - 1)p^{N+k(k)-2}$, where $p^{k(k)-1} < k \leq p^{k(k)}$, i.e., $b(k) = \lfloor \log_2 k \rfloor$.

From now on we assume that $p = 2$, $n \geq 1$ and apply Theorem A for this case. Let $g(k) = d(k) + b(k) - 2$ and $c$ denote an odd integer. Identity (1) implies $\nu_2(S(c \cdot 2^n, k)) = d(k) - 1$ for $1 \leq k \leq \min\{4, c \cdot 2^n\}$. We also set $f(k) = f_c(k) = \max\{g(k), \lfloor \log_2 (n_c(k, 2d(k)) / c) \rfloor \}$. Therefore, $c \cdot 2^{f(k)} \geq n_0(k, 2d(k))$. We note that $g(k) \leq 2[\log_2 k] - 2$. Lemma 3 in [8] yields $f(2^m) = m$ for $m \geq 1$ and $c = 1$.

In this paper we prove

Theorem 1. For all positive integers $k$ and $n$ such that $n \geq f(k)$, we have $\nu_2(k! S(c \cdot 2^n, k)) = k - 1$ or equivalently, $\nu_2(S(c \cdot 2^n, k)) = d(k) - 1$.

Numerical evidence suggests that the range might be extended for all $n$ provided $2^n \geq k$ and $c = 1$. For example, for $k = 7$, we get $g(7) = d(7) + b(7) - 2 = 4$ and $n_0(7, 2^2) = 7$; therefore by Theorem 1, if $n \geq f(7) = 4$, then $\nu_2(S(2^2, k)) = \nu_2(S(c \cdot 2^2, 7)) = 2$ for arbitrary positive integer $c$. Notice, however, that $\nu_2(S(8, 7)) = 2$ also.

We make the following

Conjecture. For all $k$ and $1 \leq k \leq 2^n$, we have $\nu_2(S(2^n, k)) = d(k) - 1$.

By Theorem 1, the Conjecture is true for all $k = 2^m$ with $m \leq n$.

In Section 3 we prove Theorem 2, which gives the exact order of $S(n, k)$ in a particular range for $k$ whose size depends on $\nu_2(n)$. Theorem 2 is the key tool in proving Theorem 1. Its proof makes use of the periodicity of the Stirling numbers. It would be interesting to determine the function $L(k)$, which is defined as the smallest integer $n'$ such that $\nu_2(S(c \cdot 2^n, k)) = d(k) - 1$ for all $n \geq n'$. By Theorem 2, we find that $L(k) \leq k - 2$ and Theorem 1 improves the upper bound on $L(k)$ if $f(k) < k - 2$.

In Section 4 we obtain some consequences of Theorem 2 by extending it for Stirling numbers of the form $S(c \cdot 2^n + u, k)$ where $u = 1, 2, \text{etc.}$ We show how to calculate $\nu_2(S(c \cdot 2^n - 1, k))$. In neither case does the order of $S(c \cdot 2^n + u, k)$ depend on $n$ (if $n$ is sufficiently large), in agreement with Theorem A.

### 3. Tools and proofs

We choose an integer $l$ such that $l \leq n$. We shall generalize identity (1) for any modulus of the form $2^l$. Observe that, for any $i$ even, $\rho_i \equiv 0 \pmod{2^l}$, and for all $i$ odd, $(-1)^{k-i}$ will have the same sign as $(-1)^{k-1}$. Therefore, by identity (1)

\begin{equation}
(2) \quad k! \ S(n, k) \equiv (-1)^{k-1} \sum_{i=1}^{k} \binom{k}{i} \rho_i \qquad \pmod{2^l}.
\end{equation}

The expression on the right-hand side of congruence (2) is called the partial Stirling number [11]. We explore identity (2) with different choices of $n$ in order to find $\nu_2(S(n, k))$.

We shall need the following

Theorem 2. Let $c$ be an odd and $n$ be a non-negative integer. If $1 \leq k \leq n + 2$ then $\nu_2(k! S(c \cdot 2^n, k)) = k - 1$, i.e., $\nu_2(S(c \cdot 2^n, k)) = d(k) - 1$. 

DIVISIBILITY BY 2 OF THE STIRLING NUMBERS OF THE SECOND KIND

Roughly speaking, Theorem 2 gives the exact value of $\nu_2\left(k!S(m, k)\right)$, for $k \geq 2$, if $m$ is divisible by $2^{k-2}$. The higher the power of 2 that divides $m$, the larger the value of $k$ that can be used. We prove Theorem 1 and then return to the proof of Theorem 2.

Proof of Theorem 1. Without loss of generality, we assume that $k > 4$. Observe that $\nu_2\left(S(c \cdot 2^n, k)\right) = d(k) - 1$ is equivalent to

$$S(c \cdot 2^n, k) \equiv 0 \pmod{2^{d(k)-1}}$$

and

$$S(c \cdot 2^n, k) \not\equiv 0 \pmod{2^{d(k)}}.$$  

The proof of identities (3) and (4) is by contradiction. To prove the former identity, we set $N = d(k) - 1$, hence Theorem A yields

$$\pi(k; 2^N) = 2^{d(k)+g(k)-3}$$

where $d(k) + b(k) - 3 < g(k) \leq f(k)$.

We assume, to the contrary of the claim, that $S(c \cdot 2^{f(k)}, k) \equiv a \neq 0 \pmod{2^N}$. By Theorem A and the period given by (5), we obtain that, for every positive integer $m \geq c$, $S(m \cdot 2^{f(k)}, k) \equiv a \neq 0 \pmod{2^N}$. This is a contradiction, for one can select $m$ so that $m \cdot 2^{f(k)}$ becomes $c \cdot 2^n$, with a large exponent $n$, and by Theorem 2, $S(c \cdot 2^n, k) \equiv 0 \pmod{2^N}$ should be for sufficiently large $n$. It follows that in fact, $S(c \cdot 2^{f(k)}, k) \equiv 0 \pmod{2^N}$, and Theorem A implies $S(c \cdot 2^n, k) \equiv 0 \pmod{2^{d(k)-1}}$ for all $n \geq f(k)$.

To derive identity (4), we set $N = d(k)$. In order to obtain a contradiction, we assume that $S(c \cdot 2^{f(k)}, k) \equiv 0 \pmod{2^N}$. Now, by Theorem A, we get $\pi(k; 2^N) = 2^{d(k)+b(k)-2}$, where $d(k) + b(k) - 2 = g(k) \leq f(k)$. We proceed in a manner similar to that used above by noting that the periodicity now yields $S(m \cdot 2^{f(k)}, k) \equiv 0 \pmod{2^N}$ for every positive integer $m \geq c$. It would imply that, for a sufficiently large $n$, $S(c \cdot 2^n, k) \equiv 0 \pmod{2^{d(k)}}$. However, this congruence contradicts Theorem 2. It follows that $S(c \cdot 2^n, k) \not\equiv 0 \pmod{2^{d(k)}}$ for $n \geq f(k)$, and the proof is now complete.

Proof of Theorem 2. We set $m = c \cdot 2^n$ and select an $l$ such that $1 \leq l \leq n+1$. By Euler’s theorem, $\phi(2^l) = 2^{l-1}$; therefore, $2^n \equiv 1 \pmod{2^l}$ if $i$ is odd. By simple summation, identity (2) yields

$$k!S(m, k) \equiv (-1)^{k-1} \sum_{i \text{ odd}} \binom{k}{i} = (-2)^{k-1} \pmod{2^l};$$

therefore, $\nu_2(k!S(m, k)) = k - 1$, provided $0 \leq k - 1 < l$.

We have two cases if $k = n + 2$. If $m$ is odd, then $n = 0$ and $k = 2$. The claim is true, since $S(m, 2) = 2^{m-1} - 1$; therefore, $\nu_2(2!S(m, 2)) = 1$. If $m$ is even, then we set $l = n + 2 \geq 3$. By induction on $l \geq 3$, we can derive that $\nu_2 \equiv 1 \pmod{2^l}$ and identity (6) is verified again.

Remark. By setting $l = n + 1$, identity (6) implies the lower bound $\nu_2(k!S(c \cdot 2^n, k)) \geq n + 1$, for $k \geq n + 2$. 

4. Related results

We will use other special cases of identity (2). Similarly to the previous proof, we get that, for all \( u \geq 0 \), \( n \geq l \geq 1 \), and \( k \leq c \cdot 2^n + u \),

\[
(7) \quad k! S(c \cdot 2^n + u, k) \equiv (-1)^{k-1} \sum_{i=1 \text{ odd}}^{k} \binom{k}{i} i^{c \cdot 2^n + u} \equiv (-1)^{k-1} \sum_{i=1 \text{ odd}}^{k} \binom{k}{i} i^u \pmod{2^l + 2}.
\]

We set

\[
h(k, u) = (-1)^{k-1} \sum_{i=1 \text{ odd}}^{k} \binom{k}{i} i^u.
\]

By identity \( x^u = \sum_{j=0}^{u} S(u, j) \binom{r}{j} j! \), we obtain

\[
h(k, u) = (-1)^{k-1} \sum_{i=1 \text{ odd}}^{k} \binom{k}{i} \sum_{j=0}^{u} S(u, j) \binom{i}{j} j! = (-1)^{k-1} \sum_{j=0}^{\min\{u, k\}} S(u, j) j! \sum_{i=1 \text{ odd}}^{k} \binom{k}{i} \binom{i}{j}.
\]

We focus on the case in which \( k > u \) and derive

\[
(8) \quad h(k, u) = (-1)^{k-1} \sum_{j=0}^{u} S(u, j) j! \binom{k}{j} \sum_{i=\text{odd}}^{k} \binom{k-j}{i-j} = (-2)^{k-1} \sum_{j=0}^{u} S(u, j) j! \binom{k}{2j}.
\]

We introduce the notation \( r(k, u) = \nu_2(h(k, u)) \). Identity (8) implies that \( r(k, u) \geq k - u - 1 \). Observe that \( |h(k, 0)| = 2^{k-1} \), and for \( u \geq 1 \),

\[
(9) \quad |h(k, u)|/2^{k-u-1} \leq \sum_{j=1}^{u} j^u 2^{u-j} k^j \leq u(2u)^u(k/2)^u = u(uk)^u.
\]

By identity (7), for \( u \geq 0 \) and any sufficiently large \( l \) and \( n \geq l \), we have \( \nu_2(k! S(c \cdot 2^n + u, k)) = r(k, u) \). In fact, \( n \geq l = r(k, u) - 1 \) will suffice; for instance, \( n \geq k - 2 \) will be large enough if \( u = 0 \) (Theorem 2). By identity (9), we derive that \( r(k, u) \leq k - u - 1 + u \log_2 k + (u + 1) \log_2 u \); therefore, \( k - u - 2 + [u \log_2 k + (u + 1) \log_2 u] \) can be chosen for \( n \) if \( u \geq 0 \). We note that, similarly to the proof of Theorem 1, this value might be decreased.

The values of \( r(k, u) \) can be calculated by identity (8). For example, if \( k > u \geq 0 \) then

\[
r(k, u) = \begin{cases} 
k - 1, & \text{if } u = 0 \\
k - 2 + \nu_2(k), & \text{if } u = 1 \\
k - 3 + \nu_2(k) + \nu_2(k + 1), & \text{if } u = 2 \\
k - 4 + 2\nu_2(k) + \nu_2(k + 3), & \text{if } u = 3 \\
k - 5 + \nu_2(k) + \nu_2(k + 1) + \nu_2(k^2 + 5k - 2), & \text{if } u = 4.
\end{cases}
\]

We state two special cases that can be proved basically differently; although, in the second case, only a partial proof comes out by the applied recurrence relations.

**Theorem 3.** For \( k \geq 2 \) and any sufficiently large \( n \), \( \nu_2(k! S(c \cdot 2^n + 1, k)) = k - 2 + \nu_2(k) \).

**Proof.** The proof follows from Theorem 2 and using the recurrence relation \( k! S(m, k) = k! \{S(m - 1, k - 1) + k! S(m - 1, k)\} \) with \( m = c \cdot 2^n + 1 \). Notice, that by Theorem 1, \( n \geq \max\{f(k), f(k - 1)\} \) will be sufficiently large.

**Theorem 4.** For \( k \geq 3 \) and sufficiently large \( n \), \( \nu_2(k! S(c \cdot 2^n + 2, k)) = k - 3 + \nu_2(k) + \nu_2(k + 1) \).
DIVISIBILITY BY 2 OF THE STIRLING NUMBERS OF THE SECOND KIND

Proof. By identity (10), we obtain \( \nu_2(k! S(c \cdot 2^n + u, k)) = r(k, 2) = k - 3 + \nu_2(k) + \nu_2(k + 1) \). Observe that \( n \geq \max \{ f(k), f(k - 1), f(k - 2) \} \) suffices.

Notice that we could have used the expansion

\[
k! S(c \cdot 2^n + 2, k) = k \{(k - 1)! S(c \cdot 2^n + 1, k - 1) + k! S(c \cdot 2^n + 1, k)\}.
\]

By Theorem 3, the first term of the second factor is divisible by a power of 2 with exponent \( k - 3 + \nu_2(k - 1) \), while the second term is divisible by 2 at exponent \( k - 2 + \nu_2(k) \). The first factor contributes an additional exponent of \( \nu_2(k) \) to the power of 2. We combine the two terms and find that there is always a unique term with the lowest exponent of 2 if \( k \not\equiv 3 \pmod{4} \). For \( k \equiv 3 \pmod{4} \), however, this argument fails short and we obtain only the lower bound \( k - 1 \) on \( \nu_2(k! S(c \cdot 2^n + 2, k)) \).

It turns out that calculating \( \nu_2(k! S(c \cdot 2^n + u, k)) \) for negative integers \( u \) is more difficult than for positive values. The periodicity guarantees that the order does not depend on \( n \) (for sufficiently large \( n \)).

We extend the function \( h(k, u) \) for negative integers \( u \). We will choose an appropriate value \( l \geq 1 \) and then set \( n \) so that it satisfies the inequality \( c \cdot 2^n + u \geq 2^l \). We use the convenient notation \( 1/i \) for the unique integer solution \( x \) of the congruence \( i \cdot x \equiv 1 \pmod{2^{l+2}} \) if \( i \) is odd. Similarly to identity (7), we obtain

\[
(11) \quad k! S(c \cdot 2^n + u, k) \equiv (-1)^{k-1} \sum_{i=1 \atop i \text{ odd}}^{k} \binom{k}{i} \left( \frac{1}{i} \right)^{u} \pmod{2^{l+2}}.
\]

For \( u < 0 \), we set

\[
h(k, u) = (-1)^{k-1} \sum_{i=1 \atop i \text{ odd}}^{k} \binom{k}{i} \left( \frac{1}{i} \right)^{-u}
\]

and express \( h(k, u) \) as a fraction \( \frac{p_k(u)}{q_k(u)} \) in lowest terms. Notice that \( \nu_2(p_k(u)) \geq k - d(k) \) holds, since \( k! \) divides both sides of (11) for any sufficiently large \( l \). The order of \( \nu_2 \left( S(c \cdot 2^n + u, k) \right) \) can be determined by choosing \( l \geq \nu_2(p_k(u)) + 1 \), and the actual order is \( \nu_2(p_k(u)) + k + d(k) \). We remark that, for \( c = 1 \), the value of \( n \) can be set to \( \nu_2(p_k(u)) \).

We focus on the case of \( u = -1 \). Let

\[
a_k = \sum_{i=1 \atop i \text{ odd}}^{k} \binom{k}{i} \frac{1}{i}.
\]

We get

\[
a_s - a_{s-1} = \binom{s}{s} \frac{1}{s} - \sum_{i=1}^{s-1} \binom{s-1}{i} \frac{1}{i} = \frac{s}{s} \left( \frac{1}{1} \right) - \sum_{i=1}^{s-1} \frac{1}{i} = \frac{s}{s} - \sum_{i=1}^{s-1} \frac{1}{i} = \frac{2^s - 2}{s}, \quad (s \geq 2).
\]

By summation, it follows that \( a_k = \sum_{i=1 \atop i \text{ odd}}^{k} \frac{k}{i} - \sum_{i=1}^{k-1} \frac{1}{i} \). Similarly, \( b_k = \sum_{i=1 \atop i \text{ even}}^{k} \binom{k}{i} \frac{1}{i} = \sum_{i=1 \atop i \text{ odd}}^{k} \frac{k}{i} \) (cf. Hietala and Winter [4], or Solution to Problem E3052, in Amer. Math. Monthly 94(1987), No. 2, p. 185). Combining these two identities, we obtain

\[
h(k, -1) = \sum_{i=1 \atop i \text{ odd}}^{k} \binom{k}{i} \frac{1}{i} = \frac{1}{2} \sum_{i=1 \atop i \text{ odd}}^{k} \frac{2^i}{i} = \frac{p_k(-1)}{q_k(-1)}.
\]

For example, for \( k = 5 \), we get \( h(5, -1) = \frac{128}{15} \), \( \nu_2(p_k(-1)) = 7 \) and \( n \geq 7 \). E.g., \( \nu_2(S(127, 5)) = \nu_2(S(255, 5)) = \cdots = 4 \). We remark that \( h(5, -1) = 4 \) holds, too. Notice that the recurrence relation \( S(N, K) = K \cdot S(N-1, K) \).
1, K) + S(N − 1, K − 1) implies that ν2(S(c · 2^n − 1, 2^n − 1) = 0 for every sufficiently large n. By the theory of p-adic numbers [6] and (12), we can derive that, for all sufficiently large n, ν2(S(c · 2^n − 1, k)) = ν2 \left( \frac{1}{2} \sum_{i=1}^{k} \frac{2^i}{q} \right) - k \cdot d(k) = ν2 \left( \frac{1}{2} \sum_{i=k+1}^{\infty} \frac{2^i}{q} \right) - k \cdot d(k) \right) \] where ν2(\alpha/\beta) is defined as ν2(\alpha)−ν2(\beta) if \alpha and \beta are integers. This fact helps us to make observations for some special cases. For instance, if n > m ≥ 3, then ν2(S(c · 2^n − 1, 2^m)) ≥ 2 holds, and, therefore, ν2(S(c · 2^n − 1, 2^m + 1)) = 1. Numerical evidence suggests that, for n > m ≥ 4, ν2(S(c · 2^n − 1, 2^m)) = 2m − 2, although we were unable to prove it.

We can determine ν2(S(c · 2^n − 1, k)) for most of the odd values of k by systematically evaluating ν2 \left( \sum_{i=1}^{k} \frac{2^{i}}{q} \right) and obtain

Theorem 5. For all sufficiently large n, ν2(S(c · 2^n − 1, k)) = d(k)−ν2(k+1), if k ≥ 1 is odd and k ≡ 5 (mod 8) and k ≡ 59 (mod 64) and k ≡ 121 (mod 128).

We leave the details of the proof to the reader.

We note that there is an alternative way of determining p_k(−1). We set

\[ I_{k-1} = \frac{k-1}{2^{k-1}} \sum_{i=1}^{k} \frac{2^i}{i}. \]

One can prove that \( I_k = \sum_{i=0}^{k} \frac{1}{(i)} \) and \( I_k = \frac{k+1}{2k} I_{k-1} + 1 \). For other properties of \( I_k \), see Comtet ([1] p. 294, Exercise 15). The latter recurrence relation simplifies the calculation of ν2(S(c · 2^n − 1, k)) for large values of k.

We can use identity (7) in a slightly different way and gain information on the structure of the sequence \{S(c · 2^n + k, S(c · 2^n + k + 1, k), · · · , S((c + 1) · 2^n + k − 1, k)) (mod 2^l)\} for every \( q, 1 ≤ q ≤ d(k) − 1 \) and sufficiently large n. We observe that the sequence always start with a one and ends with at least d(k) − q zeros. Notice that, for every \( l \) and u such that \( k + u ≥ l > k − d(k) \)

\[ 0 = k! S(u, k) ≡ (-1)^{k-1} \sum_{i=1}^{k} \binom{k}{i} r^u \mod 2^l. \]

We set \( q = l - k + d(k) \). Clearly \( 1 ≤ q ≤ d(k) − 1 \). By (7), we get that \( k! S(c · 2^n + u, k) \equiv 0 \mod 2^l \) for all \( n ≥ l - 2 ≥ 1 \). This observation yields that the \( d(k) − q \) consecutive terms,

\[ S(c · 2^n + u, k) \mod 2^l \]

are all zeros. Similarly, we can derive that \( k! S(c · 2^n + k, k) \equiv k! \mod 2^l \), i.e., \( S(c · 2^n + k, k) \equiv 1 \mod 2^l \). Identities (8) and (10) imply that there might be many more zeros in the sequence at and after the term \( S(c · 2^n, k) \mod 2^l \).

For example, if \( k = 7 \) and \( l = 5 \), then \( S(c · 2^n + u, 7) \equiv 0 \mod 2^l \), for \( u = 5 \) and \( 6 \), and all \( n ≥ 3 \). Similarly to the proof of Theorem 1, it follows that identity (13) holds if \( n ≥ f(k) \). For instance, if \( k = 23 \) and \( l = 21 \), then \( S(c · 2^n + u, 23) \equiv 0 \mod 2^l \) for \( u = 21 \) and \( 22 \) provided \( n ≥ f(23) = 7 \).

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