CHARACTERIZING THE 2-ADIC ORDER OF THE LOGARITHM

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1. INTRODUCTION

We define $v_p(x)$ as the highest power of prime p which divides the integer x. The function $v_p(x)$ is often called the p-adic order of x. In this paper we characterize the divisibility by 2 of the series $\sum_{k=1}^{\infty}(-1)^{k-1}\frac{x^k}{k}$ and $\sum_{k=1}^{\infty}\frac{x^k}{k}$, i.e., we determine their 2-adic orders. The characterization generalizes previously known results on 2-adic orders and is based on elementary proofs.

2. RESULTS

For an integer x, the p-adic order $v_p(x)$ of x is the highest power of prime p that divides x. We can think of the relations $p|x$ and $p\nmid x$ as $v_p(x)\geq 1$ and $v_p(x)=0$, respectively.

We set $v_p(0)=\infty$ and $v_p(x/y)=v_p(x)-v_p(y)$ if both x and y are integers. Therefore, for all nonzero rational numbers, the order is defined to be a finite integer. From now on, all rational numbers will be meant in lowest terms.

For rational numbers a_k $(k\geq 0)$ and rational x, the p-adic order, $v_p(\sum_{k=0}^{\infty}a_kx^k)$ of the series $\sum_{k=0}^{\infty}a_kx^k$ can be introduced as $\lim_{n\to\infty}v_p(\sum_{k=0}^{n}a_kx^k)$ if the limit exists, in which case there exists an n_0 such that $v_p(\sum_{k=0}^{n}a_kx^k)=v_p(\sum_{k=0}^{\infty}a_kx^k)$ for $n\geq n_0$. To illustrate this, we consider the series $\frac{x}{1-x}=x+x^2+x^3+\cdots$. The reader can easily verify that $v_p(\frac{x}{1-x})=v_p(x)$ if $v_p(x)\geq 1$ and the limit does not exist if $v_p(x)\leq 0$. Actually, $v_p(x+x^2+x^3+\cdots+x^n)=nv_p(x)$ if $v_p(x)<0$. Notice that if $v_2(x)=0$ then $v_2(x+x^2+x^3+\cdots+x^{2n+1})=0$, while $v_2(x+x^2+x^3\ldots+x^{2^n})\geq n$. Finding the p-adic order of functions helps in analyzing the divisibility property of the underlying or related functions. We note that Clarke [1] has recently studied the p-adic order of the logarithm by using p-adic arguments in order to characterize the divisibility properties of the *Stirling* and *partial Stirling numbers*. The interested reader should consult a book on p-adic metrics (e.g., [2]) for a general treatise of p-adic power series.

In this paper we consider the series $\log(1+x)=\sum_{k=1}^{\infty}(-1)^{k-1}\frac{x^k}{k}$ and $-\log(1-x)=\sum_{k=1}^{\infty}\frac{x^k}{k}$ and determine their 2-adic orders by elementary arguments based on binomial expansion.

In most cases the p-adic order of $\log(1+x)$ can be derived by the well-known

Theorem A (Yu [4]): We have

$$v_p(\log(1+x))=v_p\left(\sum_{k=1}^{\infty}(-1)^{k-1}\frac{x^k}{k}\right)=v_p(x)\quad\text{if } v_p(x)>\frac{1}{p-1},$$

and $v_p(\log(1+x))$ does not exist if $v_p(x)\leq 0$. In particular, for any integer x, $v_p(\log(1+x))=v_p(x)$ if $p\geq 3$ and $p|x$, or if $p=2$ and $4|x$, while for $p\nmid x$ the p-adic order $v_p(\log(1+x))$ does not exist.

In fact, Theorem A completely describes the p-adic order for $p \geq 3$. The purpose of this paper is to characterize the 2-adic orders of the two series in the case not covered by Theorem A, i.e., for every even integer x and $p=2$. We note that the proof of Theorem A is based on the observation that under the conditions of Theorem A given for p and x, the p-adic order of the terms $(-1)^{k-1}\frac{x^k}{k}$, $k \geq 2$, of the infinite series $\sum_{k=1}^{\infty}(-1)^{k-1}\frac{x^k}{k}$ exceeds that of the first term, x (cf. [2], p. 81).

If $p=2$ and $x=2$, then the following lemma (cf. [2], Ex. 7, p. 83) describes the 2-adic "behavior" of $\sum_{k=1}^{n}\frac{2^k}{k}$, i.e., the sum of the first n terms of the expansion $-\log(1-x)$.

Lemma B: The 2-adic order of the rational number $\sum_{k=1}^{n}\frac{2^k}{k}$ approaches infinity as n increases.

An elementary proof can be given based on the observation that

$$v_2\left(\sum_{k=n+1}^{\infty}\frac{2^k}{k}\right) \geq \min_{k\geq n+1}(k - v_2(k)),$$

which assures that $v_2(\sum_{k=n+1}^{\infty}\frac{2^k}{k})$ becomes arbitrarily large as $n \to \infty$. One can prove that

$$v_2\left(\sum_{k=1}^{n}\frac{2^k}{k}\right) \geq v_2\left(\sum_{k=n+1}^{\infty}\frac{2^k}{k}\right)$$

holds for infinitely many values n. In fact, a p-adic argument shows that equality holds for all n. We leave the details to the reader.

We set $v_p(\sum_{k=0}^{n}a_k x^k) = \infty$ if, for every integer $N \geq 1$, there exists an integer n_0 such that p^N divides $\sum_{k=0}^{n}a_k x^k$ for every $n \geq n_0$. In this case, $v_p(\sum_{k=0}^{n}a_k x^k) = v_p(\sum_{k=n+1}^{\infty}a_k x^k)$ holds. By the Lemma, we set $v_2(\sum_{k=1}^{\infty}\frac{2^k}{k}) = \infty$. We note that 0 and 2 play a special role in the 2-adic analysis of $\log(1-x)$ for these are the values for which $v_2(\log(1-x)) = \infty$ (cf. [2]). Our results are summarized in the following two theorems.

Theorem 1: For any even positive integer x,

$$v_2\left(\sum_{k=1}^{\infty}(-1)^{k-1}\frac{x^k}{k}\right) = \begin{cases} 2, & \text{if } x = 2, \\ 2, & \text{if } x \equiv 2 \pmod{16}, \\ 2, & \text{if } x \equiv 4 \pmod{16}, \\ 3, & \text{if } x \equiv 6 \pmod{16}, \\ 3, & \text{if } x \equiv 8 \pmod{16}, \\ 2, & \text{if } x \equiv 10 \pmod{16}, \\ 2, & \text{if } x \equiv 12 \pmod{16}, \\ v_2(x+2), & \text{if } x \equiv 14 \pmod{16}, \\ v_2(x), & \text{if } x \equiv 0 \pmod{16}. \end{cases}$$

Theorem 2: For any even positive integer x,

$$\nu_2\left(\sum_{k=1}^{\infty}\frac{x^k}{k}\right)=\begin{cases}\infty, & \text{if } x=2,\\ \nu_2(x-2), & \text{if } x\equiv 2 \pmod{16},\\ 2, & \text{if } x\equiv 4 \pmod{16},\\ 2, & \text{if } x\equiv 6 \pmod{16},\\ 3, & \text{if } x\equiv 8 \pmod{16},\\ 3, & \text{if } x\equiv 10 \pmod{16},\\ 2, & \text{if } x\equiv 12 \pmod{16},\\ 2, & \text{if } x\equiv 14 \pmod{16},\\ \nu_2(x), & \text{if } x\equiv 0 \pmod{16}.\end{cases}$$

Remark 1: The above theorems could be restated in a more compact form:

$$\nu_2\left(\sum_{k=1}^{\infty}(-1)^{k-1}\frac{x^k}{k}\right)=\begin{cases}\nu_2(x), & \text{if } x\equiv 0,4,8,12 \pmod{16},\\ \nu_2(x+2), & \text{if } x\equiv 2,6,10,14 \pmod{16},\end{cases}$$

and

$$\nu_2\left(\sum_{k=1}^{\infty}\frac{x^k}{k}\right)=\begin{cases}\nu_2(x), & \text{if } x\equiv 0,4,8,12 \pmod{16},\\ \nu_2(x-2), & \text{if } x\equiv 2,6,10,14 \pmod{16}.\end{cases}$$

Notice the sharp contrast between $\nu_2(\sum_{k=1}^{\infty}(-1)^{k-1}\frac{2^k}{k})$ and $\nu_2(\sum_{k=1}^{\infty}\frac{2^k}{k})$. We can combine the cases $x\neq 2$ of the two theorems by substituting $-x$ in place of x and carrying out the modular calculations.

For a rational $x=a/b$ with $\nu_2(x)=1$ and $b>1$, there exists a sufficiently large integer m such that $\nu_2(\log(1+x))<m$. We set $x'=a*b^{-1}$, where b^{-1} is the unique solution to the equation $b*b^{-1}\equiv 1 \pmod{2^m}$ with $0<b^{-1}<2^m$. We can proceed to determine $\nu_2(\log(1+x'))$ by Theorem 1 and observing that $\nu_2(\log(1+x))=\nu_2(\log(1+x'))$. If $x'\not\equiv 14 \pmod{16}$, then $m=4$ is an appropriate choice. However, if it turns out that the remainder is 14, then one should check whether $\nu_2(x'+2)<m$ and try a larger m if it fails. A similar method works for determining $\nu_2(\log(1-x))$, too.

For example, if $x=6/5$, then $\nu_2(\log(1-6/5))=2$ follows easily with $m=4$. We use $m=5$ and have $x'=6*13\equiv 14 \pmod{16}$ in order to obtain $\nu_2(\log(1+6/5)=\nu_2(6*13+2)=4$. For $x=426/555$, we start with $m=4$. Since $x'=426*3\equiv 14 \pmod{16}$ and $\nu_2(426*3+2)=8$, we note that we need a larger m. By using $m=10$, we obtain $x'=426*131\equiv 14 \pmod{16}$ and $\nu_2(\log(1+426/555))=\nu_2(426*131+2)=9$.

Remark 2: Similarly to the proof of Theorem A, we observe that $\nu_2(2^s)<\nu_2((2^s)^k/k)$ if $k\geq 2$ and $s\geq 2$. Therefore,

$$\nu_2\left(\sum_{k=1}^{\infty}\frac{(2^s)^k}{k}\right)=\nu_2\left(\sum_{k=1}^{\infty}(-1)^{k-1}\frac{(2^s)^k}{k}\right)=\nu_2(2^s)=s \text{ if } s\geq 2.$$

3. PROOFS

Proof of Theorem 1: The case of $x = 2$ is easily verified by checking the first couple of terms of $\sum_{k=1}^{\infty}(-1)^{k-1}\frac{x^k}{k}$. Indeed, $v_2(\sum_{k=1}^{4}(-1)^{k-1}\frac{2^k}{k}) = 2$ and $v_2(2^k/k) > 2$ for $k \geq 5$.

If $x = 6$ or 10, then by inspecting the sum of the first few terms we obtain, similarly to the case of $x = 2$, that the orders are 3 and 2, respectively.

We can extend these results for $x \equiv 2, 6,$ and $10 \pmod{16}$. From now on a denotes an arbitrary integer while b is an arbitrary odd integer. The basic idea is that if $v_2(\sum_{k=1}^{\infty}(-1)^{k-1}\frac{x^k}{k}) = r < s$ then $v_2(\sum_{k=1}^{\infty}(-1)^{k-1}\frac{(x+a2^s)^k}{k}) = r$, too, since $x^k \equiv (x+a2^s)^k \pmod{2^s}$. [Of course, the same applies if we omit the factors $(-1)^{k-1}$.] By the previous observations, we can set $s = 4$.

For $x \equiv 0, 4, 8,$ or $12 \pmod{16}$, the statement follows from Theorem A which claims that the order must be $v_2(x)$.

Instead of simply proving the remaining case $x \equiv 14 \pmod{16}$, we combine the cases $x \equiv 2$ and $14 \pmod{16}$ to make this proof transparent to prove Theorem 2. Let $s = 4$. We calculate the 2-adic order of $\sum_{k=1}^{\infty}(-1)^{k-1}\frac{x^k}{k}$ using the binomial expansion of the terms $x^k = (b2^s + 2c)^k$ where c is either 1 or -1. The expansion yields

$$(b2^s + 2c)^k = (2(b2^{s-1} + c))^k = \sum_{\ell=0}^{k} 2^k \binom{k}{\ell} (b2^{s-1})^{\ell} c^{k-\ell}.$$

Note that the identity $\binom{k}{\ell} = \frac{k}{\ell}\binom{k-1}{\ell-1}$ implies that $\binom{k}{\ell}/k$ is an integer multiple of $1/\ell$. Consider the sum

$$\sum_{k=1}^{\infty}(-1)^{k-1}\frac{(b2^s+2c)^k}{k}$$

in three terms, one term for $\ell = 0$, another for $\ell = 1$, and the last one for all the remaining cases, $\ell \geq 2$. We get

$$\begin{aligned}&\sum_{k=1}^{\infty}(-1)^{k-1}\frac{(b2^s+2c)^k}{k} \\ &= -\sum_{k=1}^{\infty}\frac{(-2c)^k}{k} + \sum_{k=1}^{\infty} b2^{k+s-1}(-c)^{k-1} + \sum_{k=1}^{\infty}(-1)^{k-1}\sum_{\ell=2}^{k}\frac{\binom{k-1}{\ell-1} b^{\ell} 2^{\ell(s-1)+k} c^{k-\ell}}{\ell}.\end{aligned} \tag{1}$$

Obviously, the 2-adic order of the second term is s if $b \neq 0$. Notice that the third term is always divisible by 2^{s+1} for $s \geq 3$, since this condition implies that $\ell(s-1) + k - v_2(\ell) \geq \ell(s-1) + k - \log_2 \ell \geq s+1$. It turns out that the 2-adic order of the first term on the right side of identity (1) is 2 if $c = 1$ as we have seen it at the beginning of the proof. By Lemma B, the 2-adic order of the first term is ∞ if $c = -1$. It follows that $v_2(\sum_{k=1}^{\infty}(-1)^{k-1}(b2^s + 2c)^k/k) = s$ if $c = -1$ (and $b \neq 0$), while it is 2 if $c = 1$. □

Proof of Theorem 2: Basically, the proof of Theorem 1 can be repeated here except for $x = 2$, which case is the content of Lemma B. Careful inspection reveals that the 2-adic orders are switched for $x \equiv 6$ and $10 \pmod{16}$.

Similarly to identity (1), we have

$$\sum_{k=1}^{\infty}\frac{(b2^s+2c)^k}{k}=\sum_{k=1}^{\infty}\frac{(2c)^k}{k}+\sum_{k=1}^{\infty}b2^{k+s-1}c^{k-1}+\sum_{k=1}^{\infty}\sum_{\ell=2}^{k}\frac{\binom{k-1}{\ell-1}b^{\ell}2^{\ell(s-1)+k}c^{k-\ell}}{\ell}, \tag{2}$$

where the last term is always divisible by 2^{s+1} for $s \geq 3$.

By simply switching the cases $c=1$ and $c=-1$ in the previous proof and using identity (2), we derive that $v_2(\sum_{k=1}^{\infty}\frac{(b2^s+2c)^k}{k})=s$ if $c=1$ (and $b \neq 0$), while it is 2 if $c=-1$. □

We note that Clarke [1] has recently proved similar results by using p-adic arguments.

Lemma B points to the odd behavior of $v_2(\sum_{k=1}^{n}\frac{x^k}{k})$ at $x=2$. Analysis of this behavior gives rise to the question on the rate at which $v_2(\sum_{k=1}^{n}\frac{2^k}{k})$ increases as n gets larger. We were unable to answer this question; however, numerical evidence suggests some pattern for the increase of the 2-adic order. The following conjecture has been proposed in [3], in the context of the divisibility by 2 of the Stirling numbers of the second kind, $S(a2^n-1, 2^m)$, where $n>m\geq 4$ and a is a positive integer.

Conjecture 3: For $m \geq 4$, $v_2(\sum_{k=1}^{2^m}\frac{2^k}{k})=2^m+2m-2$.

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