

The Sprague-Grundy function of the *real* game Euclid

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Abstract

The game *Euclid*, introduced and named by Cole and Davie, is played with a pair of nonnegative integers. The two players move alternatively, each subtracting a positive integer multiple of one of the integers from the other integer without making the result negative. The player who reduces one of the integers to zero wins. Unfortunately, the name *Euclid* has also been used for a subtle variation of this game due to Grossman in which the game stops when the two entries are equal. For that game, Straffin showed that the losing positions (a, b) with $a < b$ are precisely the same as those for Cole and Davie's game. Nevertheless, the Sprague-Grundy functions are not the same for the two games. We give an explicit formula for the Sprague-Grundy function for the original game of Euclid and we explain how the Sprague-Grundy functions of the two games are related.

1. Introduction

Euclid is a two person impartial combinatorial game, introduced and named by Cole and Davie [1]. It starts with a pair of positive integers. The players move alternatively, each subtracting a positive integer multiple of one of the integers from the other integer without making the result negative. The player who reduces one of integers to zero wins. It was shown in [1] that for $a < b$, the position (a, b) is a losing position if and only if $b < \phi a$, where $\phi = \frac{\sqrt{5}+1}{2}$ is the Golden ratio. Aspects of Euclid were studied in [12]. Unfortunately, there is a common misunderstanding concerning *Euclid*.

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9	0	9	4	3	1	1	0	0	0	1	
8	0	8	4	2	2	0	0	0	1	0	
7	0	7	3	1	1	0	0	1	0	0	
6	0	6	3	2	0	0	1	0	0	0	
5	0	5	1	1	0	1	0	0	0	1	
4	0	4	2	0	1	0	0	1	2	1	
3	0	3	0	1	0	1	2	1	2	3	
2	0	2	1	0	2	1	3	3	4	4	
1	0	1	2	3	4	5	6	7	8	9	
0		0	0	0	0	0	0	0	0	0	
a	b	0	1	2	3	4	5	6	7	8	9

Table 1: Sprague-Grundy values $\mathcal{G}(a, b)$ for $a, b \leq 9$

Grossman [5] introduced a subtle variation of Euclid in which the game stops when the two entries are equal. Notice that *Grossman's game is just the misère version of Euclid*. In the literature, the term *Euclid* commonly refers to Grossman's variation and various aspects and extensions of this game have been studied in [8, 9, 2, 4, 3]. In particular, the misère version of Grossman's game was studied in [6]. In this paper we will reserve the term *Euclid* for Cole and Davie's original game, and refer to its popular variation as *Grossman's game*. For Grossman's game, Straffin [13] showed that the losing positions (a, b) with $0 < a < b$ are precisely the same as those for Euclid. Nevertheless, the Sprague-Grundy functions are not the same for the two games. For a position (a, b) in the original game of Euclid, we denote its Sprague-Grundy value $\mathcal{G}(a, b)$, while for Grossman's game, we denote it $\mathcal{G}_G(a, b)$. Nivasch [11] proved that $\mathcal{G}_G(a, b) = \lfloor \lfloor \frac{b}{a} - \frac{a}{b} \rfloor \rfloor$. Table 1 gives the Sprague-Grundy values of position (a, b) for $a, b \leq 9$, and in Figure 1, the possible moves are shown for positions with $a \leq b \leq 5$. The analogous information is given for Grossman's game in Table 2 and Figure 2.

In order to present the formula for $\mathcal{G}(a, b)$, we use that the continued fraction expansion $[a_0, a_1, \dots, a_n]$ of b/a ,

$$\frac{b}{a} = a_0 + \frac{1}{a_1 + \frac{1}{a_2 + \frac{1}{\ddots + \frac{1}{a_{n-1} + \frac{1}{a_n}}}}},$$

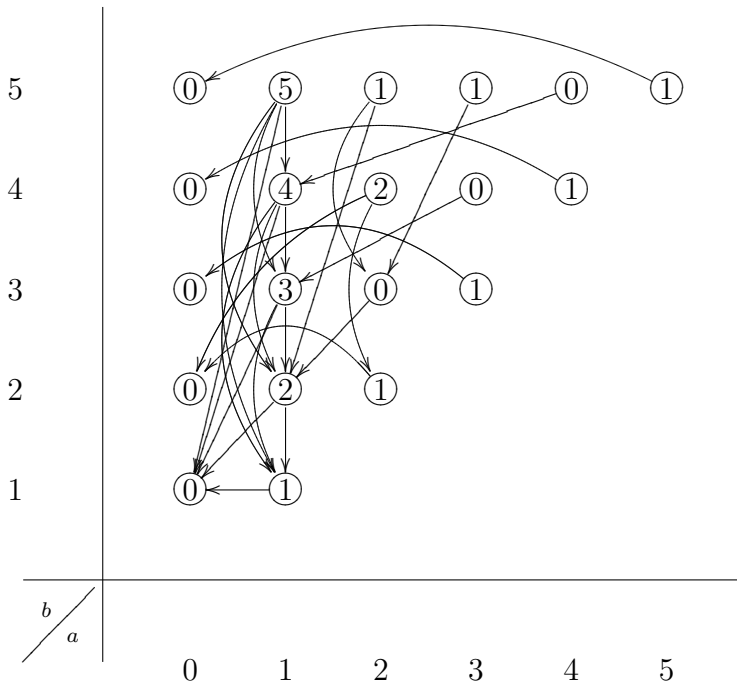


Figure 1: The moves in Euclid for $a \leq b \leq 5$

and we adopt the convention that $a_n > 1$ if $n > 0$.

Theorem 1. *Let $0 < a < b$, consider the continued fraction expansion $[a_0, a_1, \dots, a_n]$ of b/a , and let $\mathcal{I}(a, b)$ be the largest nonnegative integer i such that*

$$a_0 = \dots = a_{i-1} \leq a_i.$$

Then the Sprague-Grundy value of the position (a, b) in the game Euclid is

$$\mathcal{G}(a, b) = \left\lfloor \frac{b}{a} \right\rfloor - \begin{cases} 0 & : \text{if } \mathcal{I}(a, b) \text{ is even,} \\ 1 & : \text{otherwise.} \end{cases}$$

It is natural to ask whether the above result has a formulation that doesn't involve continued fractions, analogous to Nivasch's formula for Grossman's game [11]. The following corollary and Theorem 3 (except for a special case) provide such a formulation. We remark that when $a_0 = 1$, the number $\lambda_1 = \phi$, the Golden ratio, and so the corollary extends Cole and Davie's determination of the losing positions.

9	8	4	2	1	1	0	0	0	0	
8	7	3	2	1	0	0	0	0	0	
7	6	3	1	1	0	0	0	0	0	
6	5	2	1	0	0	0	0	0	0	
5	4	2	1	0	0	0	0	0	1	
4	3	1	0	0	0	0	1	1	1	
3	2	0	0	0	1	1	1	2	2	
2	1	0	0	1	2	2	3	3	4	
1	0	1	2	3	4	5	6	7	8	
a	b	1	2	3	4	5	6	7	8	9

Table 2: Sprague-Grundy values $\mathcal{G}_G(a, b)$ for $a, b \leq 9$

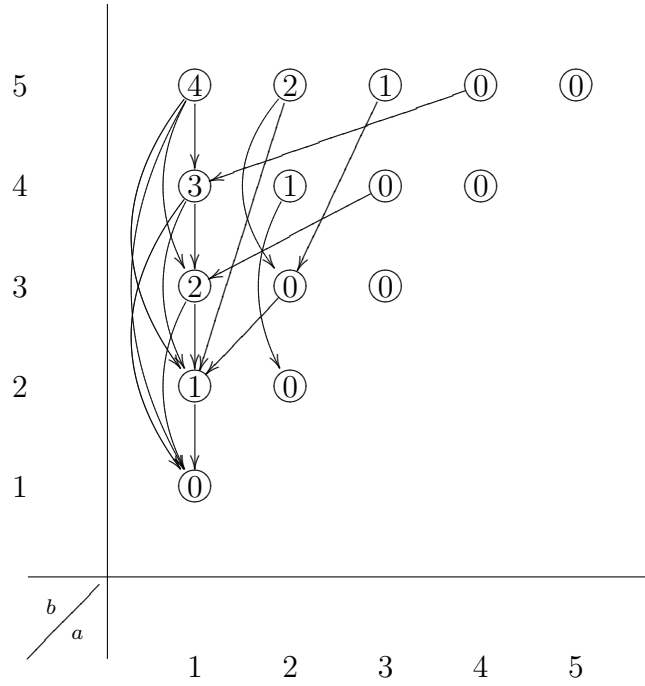


Figure 2: The moves in Grossman's game for $a \leq b \leq 5$

Corollary. If $0 < a < b$, let $a_0 = \lfloor \frac{b}{a} \rfloor$ and set

$$\lambda_1 = \frac{a_0 + \sqrt{a_0^2 + 4}}{2}, \lambda_2 = \frac{a_0 - \sqrt{a_0^2 + 4}}{2} \text{ and } x_n = \frac{\lambda_1^{n+2} - \lambda_2^{n+2}}{\lambda_1^{n+1} - \lambda_2^{n+1}},$$

for $n = 0, 1, 2, \dots$. If $a_0 > 1$ and $b/a = x_n$ for some $n \in \mathbb{N}$, then

$$\mathcal{G}(a, b) = \begin{cases} a_0 & : \text{if } \frac{b}{a} < \lambda_1, \\ a_0 - 1 & : \text{otherwise.} \end{cases}$$

while if $a_0 = 1$ or b/a is not equal to x_n for any $n \in \mathbb{N}$, then

$$\mathcal{G}(a, b) = \begin{cases} a_0 & : \text{if } \frac{b}{a} > \lambda_1, \\ a_0 - 1 & : \text{otherwise.} \end{cases}$$

Notice that from the above corollary and Nivasch's formula for $\mathcal{G}_G(a, b)$, we see that $\mathcal{G}(a, b)$ and $\mathcal{G}_G(a, b)$ differ by at most one. In fact, one has:

Theorem 2. For $0 < a < b$, suppose that b/a has continued fraction expansion $[a_0, a_1, \dots, a_n]$, and that b'/a' is the rational number with continued fraction expansion $[a_0, a_1, \dots, a_n + 1]$. Then $\mathcal{G}(a, b) = \mathcal{G}_G(a', b')$.

Remark 1. Theorem 1 may be regarded as an extension of [8, Theorem 1] and [7, Theorem 3], which give the losing positions of Grossman's game in terms of continued fractions. Euclid can be played on the Stern-Brocot and Calkin-Wilf tree in the same way that Grossman's game is treated in [9] and [7] respectively.

Remark 2. If one identifies the positions in the game Euclid with the corresponding continued fraction expansions $[a_0, a_1, \dots, a_n]$, then each move amounts to a reduction in the actual first term. Regarding the integers a_i as numbers of counters, one can think of Euclid as a game of Nim in which one may only take counters from the leftmost pile. This game was studied by K.T. Tan [14] and reinvented by Lionel Levine [10] who called it *Serial Nim*. Thus Euclid is equivalent to Serial Nim. In particular, Theorem 1 can be deduced directly from [10, Prop. 5.1] which follows by the continued fraction based method outlined in [9].

Remark 3. Theorems 1 and 2 show that Grossman's game $G_G(a, b)$ can be read from the continued fraction expansion of b/a . Indeed, if b/a has continued fraction expansion $[a_0, a_1, \dots, a_n]$, then one obtains

$$G_G(a, b) = \left\lfloor \left\lfloor \frac{b}{a} - \frac{a}{b} \right\rfloor \right\rfloor = \left\lfloor \frac{b}{a} \right\rfloor - \begin{cases} 0 & : \text{if } \mathcal{I}(a, b) \text{ is even,} \\ 1 & : \text{otherwise,} \end{cases}$$

except in the special case where $a_0 = a_1 = \dots = a_n$, in which special case,

$$G_G(a, b) = \left\lfloor \left\lfloor \frac{b}{a} - \frac{a}{b} \right\rfloor \right\rfloor = \left\lfloor \frac{b}{a} \right\rfloor - \begin{cases} 0 & : \text{if } \mathcal{I}(a, b) \text{ is odd,} \\ 1 & : \text{otherwise.} \end{cases}$$

Combining Theorem 1, Nivasch's formula and Remark 3, we deduce:

Theorem 3. *For $0 < a \leq b$, suppose that b/a has continued fraction expansion $[a_0, a_1, \dots, a_n]$. Then $\mathcal{G}(a, b) = \left\lfloor \left\lfloor \frac{b}{a} - \frac{a}{b} \right\rfloor \right\rfloor$ unless $a_0 = a_1 = \dots = a_n$, in which special case, $\mathcal{G}(a, b) = \left\lfloor \left\lfloor \frac{b}{a} - \frac{a}{b} \right\rfloor \right\rfloor + (-1)^n$.*

2. The proof of Theorems 1 and 2

Notice that in both Euclid and Grossman's game, the moves do not alter the GCD of the entries of the positions. It follows that we may assume without loss of generality that the GCD is one. Thus, the position (a, b) is completely determined by the fraction b/a , or equivalently by the continued fraction expansion of b/a . So, in the proofs we give below, we identify the positions with their associated continued fraction expansion. Notice that in both games, from a position $[a_0, a_1, \dots, a_n]$ with $n > 0$, there are a_0 possible moves:

$$[a_0, a_1, \dots, a_n] \mapsto [a_0 - i, a_1, \dots, a_n], \text{ for } 1 \leq i < a_0,$$

and

$$[a_0, a_1, \dots, a_n] \mapsto [a_1, \dots, a_n].$$

From the position $[a_0]$ with $a_0 > 0$, there are a_0 possible moves in Euclid, $[a_0] \mapsto [a_0 - i]$ with $1 \leq i \leq a_0$, but in Grossman's game only the moves with $i < a_0$ are permitted.

Proof of Theorem 2. Let S denote the set of finite continued fractions and let $S_G = S \setminus \{[0]\}$. So we may identify S (resp. S_G) with the set of positions

with GCD one in the game of Euclid (resp. Grossman's game). Consider the map $\sigma : S \rightarrow S_G$ defined by

$$\sigma : [a_0, a_1, \dots, a_n] \mapsto [a_0, a_1, \dots, a_n + 1].$$

The map σ clearly commutes with the possible moves. Moreover, notice that the terminal position in Euclid is $[0]$, while in Grossman's game, the terminal position is $[1]$. So σ respects the terminal positions. The image $\sigma(S)$ is the set of continued fractions $[a_0, a_1, \dots, a_n]$ where $a_n \geq 3$ for $n > 0$ and $a_0 \geq 1$ for $n = 0$. Notice that from every position in $\sigma(S)$, all the possible moves lead to positions in $\sigma(S)$. Thus σ is a game isomorphism from S to $\sigma(S)$. It follows that the Sprague-Grundy value in Grossman's game of $\sigma([a_0, a_1, \dots, a_n])$ is the same as the Sprague-Grundy value in Euclid of $[a_0, a_1, \dots, a_n]$. This establishes Theorem 2. \square

Theorem 1 can be established in several ways. One could use Theorem 2 and adapt ideas from [11]. Alternately, one could induct on the length of the continued fractions, using ideas from [8]. Instead, we prefer to give a direct, self-contained proof.

Proof of Theorem 1. Let \mathcal{I} and \mathcal{G} be the functions defined in the statement of Theorem 1; by abuse of language, we will write $\mathcal{I}(p)$ and $\mathcal{G}(p)$ for their values at a position $p = [a_0, a_1, \dots, a_n]$. We must establish the following two defining properties:

1. For every move $p \mapsto q$, we have $\mathcal{G}(q) \neq \mathcal{G}(p)$.
2. If $\mathcal{G}(p) > 0$, then for all integers k with $0 \leq k < \mathcal{G}(p)$, there exists a move $p \mapsto q$ such that $\mathcal{G}(q) = k$.

In the following we will make repeated use of the following obvious fact: if $p = [a_0, a_1, \dots, a_n]$ and $\mathcal{I}(p)$ is odd, then $a_0 \leq a_1$; indeed, if $a_0 > a_1$, then we would have $\mathcal{I}(p) = 0$. Similarly, if $\mathcal{I}(p)$ is even then $a_0 \geq a_1$.

To establish (1), suppose we have a move $p \mapsto q$ with $\mathcal{G}(q) = \mathcal{G}(p)$. First suppose that $q = [a_0 - i, a_1, \dots, a_n]$ for some $1 \leq i < a_0$. From the definition of \mathcal{G} , it is clear that necessarily $i = 1$, $\mathcal{I}(p)$ is odd and $\mathcal{I}(q)$ is even. As $\mathcal{I}(p)$ is odd, $a_0 \leq a_1$, and as $\mathcal{I}(q)$ is even, $a_0 - 1 \geq a_1$. Hence $a_0 \leq a_1 \leq a_0 - 1$, which is impossible. So we may assume that $q = [a_1, \dots, a_n]$. At first sight, as $\mathcal{G}(q) = \mathcal{G}(p)$, there are three possibilities:

- (i) $a_0 = a_1 - 1$ and $\mathcal{I}(p)$ is even and $\mathcal{I}(q)$ is odd,

- (ii) $a_0 = a_1 + 1$ and $\mathcal{I}(p)$ is odd and $\mathcal{I}(q)$ is even,
- (iii) $a_0 = a_1$ and $\mathcal{I}(p)$ and $\mathcal{I}(q)$ have the same parity.

But case (i) is impossible, since $a_0 \geq a_1$ when $\mathcal{I}(p)$ is even, case (ii) is impossible since $a_0 \leq a_1$ when $\mathcal{I}(p)$ is odd, and case (iii) is in contradiction with the definition of \mathcal{I} .

To establish (2), suppose that $0 \leq k < \mathcal{G}(p)$. First suppose that $\mathcal{I}(p)$ is odd, so $\mathcal{G}(p) = a_0 - 1$. Consider the position $q = [k + 1, a_1, \dots, a_n]$. Since $\mathcal{I}(p)$ is odd, $a_0 \leq a_1$. In particular, $k + 1 < a_1$ and thus $\mathcal{I}(q) = 1$. It follows that $\mathcal{G}(q) = k$, as required. So it remains to treat the case where $\mathcal{I}(p)$ is even. In this case, $\mathcal{G}(p) = a_0$ and $a_0 \geq a_1$.

We first treat the situation where $k = 0$. Assume for the moment that $a_0 > 1$. Consider $q = [1, a_1, \dots, a_n]$. Notice that we may assume that $\mathcal{I}(q)$ is even, since otherwise $\mathcal{G}(q) = 0$, as required. In particular, we have $a_1 = 1$. Let $q' = [a_1, \dots, a_n]$. But if $\mathcal{I}(q)$ is even, then $\mathcal{I}(q')$ is odd and hence $\mathcal{G}(q') = a_1 - 1 = 0$, as required. Similarly, if $a_0 = 1$, then as $\mathcal{I}(p)$ is even, we have $a_1 = 1$, and since $\mathcal{I}(p)$ is even, $\mathcal{I}(q')$ is odd and $\mathcal{G}(q') = 0$. This completes the case $k = 0$.

Now suppose that $0 < k < \mathcal{G}(p)$ and let $q = [k, a_1, \dots, a_n]$. If $\mathcal{I}(q)$ is even, then $\mathcal{G}(q) = k$, as required. So we may assume that $\mathcal{I}(q)$ is odd and thus $k \leq a_1$. In this case, we have $\mathcal{G}(q) = k - 1$. Let $q' = [k + 1, a_1, \dots, a_n]$. If $\mathcal{I}(q')$ is odd, then $\mathcal{G}(q') = k$, as required, so we may assume that $\mathcal{I}(q')$ is even, and therefore $k + 1 \geq a_1$. Thus $k + 1 \geq a_1 \geq k$. Hence, either $k + 1 = a_1$ or $k = a_1$. Consider $q'' = [a_1, \dots, a_n]$. If $k = a_1$, then as $\mathcal{I}(q)$ is odd, $\mathcal{I}(q'')$ is even, and hence $\mathcal{G}(q'') = a_1 = k$, as required. Finally, if $k + 1 = a_1$, then as $\mathcal{I}(q')$ is even, $\mathcal{I}(q'')$ is odd, and hence $\mathcal{G}(q'') = a_1 - 1 = k$, as required. \square

3. The proof of the Corollary

Let $0 < a < b$ and suppose that b/a has continued fraction expansion $[a_0, a_1, \dots, a_n]$. So $a_0 = \lfloor \frac{b}{a} \rfloor$. First suppose that the a_i , for $i = 0, \dots, n$, are not all equal. Let λ_1 denote the number with constant infinite continued fraction expansion $[a_0, a_0, a_0, \dots]$; one easily verifies that $\lambda_1 = \frac{a_0 + \sqrt{a_0^2 + 4}}{2}$. It is well known and easy to see that $b/a > \lambda_1$ if and only if $\mathcal{I}(a, b)$ is even. So Theorem 1 gives

$$\mathcal{G}(a, b) = \begin{cases} a_0 & : \text{if } \frac{b}{a} > \lambda_1, \\ a_0 - 1 & : \text{otherwise.} \end{cases}$$

Now consider the case where the a_i , for $i = 0, \dots, n$, are all equal. Since our convention is that $a_n > 1$, we have $a_0 > 1$. Notice that by definition, $\mathcal{I}(a, b)$ is even if and only if n is even. Let x_i denote the rational number with continued fraction expansion $[a_0, a_1, \dots, a_i]$. Writing $x_i = q_{i+1}/q_i$, we have by induction

$$x_{i+1} = a_0 + \frac{1}{x_i} = \frac{a_0 q_{i+1} + q_i}{q_{i+1}},$$

and so the q_i verify the recurrence relation $q_{i+2} = a_0 q_{i+1} + q_i$, with $q_0 = 1, q_1 = a_0$. Solving this recurrence relation gives

$$q_i = \frac{\lambda_1^{i+1} - \lambda_2^{i+1}}{\lambda_1 - \lambda_2},$$

where $\lambda_1 = \frac{a_0 + \sqrt{a_0^2 + 4}}{2}$, $\lambda_2 = \frac{a_0 - \sqrt{a_0^2 + 4}}{2}$ and so

$$x_n = \frac{\lambda_1^{n+2} - \lambda_2^{n+2}}{\lambda_1^{n+1} - \lambda_2^{n+1}}.$$

Note that if n is odd, λ_2^{n+1} is positive and so

$$\begin{aligned} \lambda_1 > \lambda_2 &\implies \lambda_1 \lambda_2^{n+1} > \lambda_2^{n+2} \\ &\implies \lambda_1 (\lambda_1^{n+1} - \lambda_2^{n+1}) < \lambda_1^{n+2} - \lambda_2^{n+2} \\ &\implies \lambda_1 < \frac{\lambda_1^{n+2} - \lambda_2^{n+2}}{\lambda_1^{n+1} - \lambda_2^{n+1}} = x_n. \end{aligned}$$

Similarly, if n is even, $\lambda_1 > x_n$. Thus, as $\mathcal{I}(a, b)$ is even if and only if n is even, Theorem 1 gives

$$\mathcal{G}(a, b) = \begin{cases} a_0 & : \text{if } n \text{ is even (i.e., if } x_n = \frac{b}{a} < \lambda_1), \\ a_0 - 1 & : \text{otherwise.} \end{cases}$$

□

Remark 4. Notice that deciding whether $a/b = x_n$ for some n is easy; it is visible from the continued fraction expansion of b/a , as we saw in the above proof.

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