On approximating point spread distributions

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On approximating point spread distributions

Tamás Lengyel*  
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Abstract

We discuss some properties of the point spread distribution, defined as the distribution of the difference of two independent binomial random variables with the same parameter $n$ including exact and approximate probabilities and related optimization issues. We use various approximation techniques for different distributions, special functions, and analytic, combinatorial and symbolic methods, such as multi-summation techniques. We prove that in case of unequal success rates, if these rates change with their difference kept fixed and small, and $n$ is appropriately bounded, then the point spread distribution only slightly changes for small point differences. We also prove that for equal success rates $p$, the probability of a tie is minimized if $p = 1/2$. Numerical examples are included for the case with $n = 12$.

1. Introduction

Questions regarding the point spread distribution in certain sports present a rich variety of interesting problems. In high-scoring sports, e.g. basketball, the underlying scoring distributions can be modeled by independent binomial random variables, while in low-scoring sports, e.g. baseball, hockey and soccer, independent Poisson random variables might be used. Point spreads are often used to set fairly equal winning odds in terms of scoring differences in matches between players or teams of widely different strengths. In fact, bookmakers set a point spread through the betting odds for teams playing to beat projected point spreads. In terms of scoring differences, this is achieved by giving players in 'upset' games, respectively. We assume that $X$ and $Y$ are independent binomially distributed random variables and introduce the probability $f(d, n, p, ε)$ of a tie and that of an exact point spread $d$, $f(d, n, p, ε) = f(d+1, n, p, ε)$.

We study the probability $f(0, n, p, ε) = f(n, p, ε)$ of a tie and that of an exact point spread $d$, $f(d, n, p, ε) = f(d+1, n, p, ε)$.

Keywords: Skellam distribution; approximating distributions; asymptotic enumeration; special functions; multi-summation

AMS Subject Classification: 62E17; 05A16; 33F10

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Theorem 2.1. Let \( F(n, p, \varepsilon) \) denote the probability that in \( n \) trials, the stronger player wins.

Now we prove an interesting symmetry property.

**2. Symmetry**

For \( n = 12 \) and \( \varepsilon = 0 \), we include a table showing the probabilities for various values of \( d \) and \( p \). The table reveals interesting patterns in the distribution.

Example

Players 1 and 2 are two basketball players. Player 1 makes 65% of his free-throws, while Player 2 is even better and makes 75%. They have a contest in which they each shoot 12 free-throws. You can assume that the scoring luck is totally independent of the previous or current scoring experience of either player. We are interested in the following probabilities.

A) What is the probability that this free-throw contest will end in a tie?
B) What is the probability that Player 2 will win?
C) What are the chances that, say, Player 2 scores two more than Player 1?
D) How about at least two more?

The answers can be calculated easily with most statistical software, and perhaps in the most economical way using S-PLUS, at least in terms of the necessary code. In fact, the answers are:

(A) 0.1533
(B) 0.6249
(C) 0.1676
(D) 0.4482

Other values related to the point spread distribution can be derived similarly.

We focus on questions (B) and (D), in particular. After calculating these values, we tested similar settings within a given range but with the same difference in shooting success rates and found, somewhat surprisingly, that the answers obtained changed only slightly in (B) and (D). This suggests that the point spread distribution will change only a very small degree when the success rates are close to one. We extend some of the calculations to negative values of \( \varepsilon \).

2. Symmetry

Theorem 2.1

Let \( f_d(n, p, \varepsilon) \) denote the probability that, in \( n \) trials, the 'stronger' player with success rate \( p \) accumulates at least \( d \) more points than the 'weaker' player with success rate \( p - \varepsilon \). The function \( f_d(n, p, \varepsilon) \) is symmetric about \( (1 + \varepsilon)/2 \) for every \( n \geq 1 \).
If $\frac{\partial}{\partial \beta}$ the probability of a tie is $d = \beta$ is $\beta$.

Remark 1

Sometimes, we use the short notation $f(d, \ldots, u)$ of the polynomial $f$ in Equation (1), it can be proved that the leading coefficient of $f$ in Equation (1) shows that $d$.

Remark 2

The definition of $f$ in Equation (1) shows that $d$.

Theorem 2. The claim is that $f(d, \ldots, u)$ is $d$.

Proof. We observe that

\[
\prod_{d=1}^{d} (d - 1)^{d} f(d, \ldots, u) = (d - 1)^{d} f(d, \ldots, u) = f(d, \ldots, u)
\]
For instance, if we obtain the following theorem.

Theorem: If we obtain the following theorem.

\[ P(\theta + d - 1, \theta, \varepsilon) \leq \sum_{n=0}^{\infty} \frac{\mu^n}{\varepsilon(\theta - 1)} \approx \left( \frac{\theta}{\theta + 1} \right)^{1/2} P(\theta, \theta, \varepsilon) \]

For a small \( \theta > 0 \), we can also approximate the above approach for any difference \( \theta > 0 \) by using the hypergeometric function. (c f [2].) If \( \varepsilon \) is small, \( \varepsilon > 0 \) may also be approximated by (1.13.2).

\[ x < \frac{\theta}{\theta + 1} \]

\[ \frac{x}{\varepsilon} \approx \frac{x}{\varepsilon - 1} \]

By using the hypergeometric function, \( \varepsilon \) may also be approximated by (1.13.2).

we can generalize the above approach for any difference \( \theta > 0 \) by using the hypergeometric function. (c f [2].) If \( \varepsilon \) is small, \( \varepsilon > 0 \) may also be approximated by (1.13.2).
Assume that in the shooting competition the players shoot in an alternating fashion. Let \( X \) and \( Y \) denote the number of successful shots made by Players 2 and 1, respectively, and \( \epsilon > 0 \) if the stronger or the other player scores, respectively, and stays in place if either both or none of the players score. We can restrict our attention to the range of the approximation for \( q < \epsilon \) for both Players 2 and 1, respectively. For \( \epsilon > 0 \) we have the following.

\[
P(X = k) = \frac{1}{\sqrt{2\pi \epsilon}} \int_{X = k \pm \frac{1}{2}}^{} \exp\left(-\frac{(x - \mu)^2}{2\epsilon}\right) \, dx
\]

for \( X \) and \( \epsilon \). This particular location gives either a local minimum or maximum.

5. Random walk approach

As we observed above, we can restrict our attention to the range of the approximation for \( q < \epsilon \). In another example, we can get a good approximation for the probability of a tie to \( Wagen [5] \). The author conjectures that for some cases though, it might be helpful.

For example, coefficient extraction of the (Laurent) power series expansion \( (\epsilon) \) stands for the coefficient of the term \( \epsilon^k \) in the (Laurent) power series expansion \( (n, p, \epsilon) \).

\[
\sum_{k} g_{n} f_{k} d \sim \frac{\mu \sqrt{\pi(\mu \epsilon)} \epsilon}{\epsilon + 1} \sim \frac{1}{\sqrt{2\pi \epsilon}} \int_{X = \mu \pm \frac{1}{2}}^{} \exp\left(-\frac{(x - \mu)^2}{2\epsilon}\right) \, dx
\]

which looks simpler than the one given in Section 3. For example, in this way we get the approximate answer 0.1542 to (A). Note the difference in the rate of decrease for extracting coefficients. In some cases though, it might be helpful.
\[
0 \geq \left( \frac{z}{\sqrt{\varepsilon}} + \left( \frac{z}{\sqrt{\varepsilon} + 1} - d \right) \left( \sqrt{\varepsilon} + 1 \right) \right) \left( \frac{u_0^2 \varepsilon^3 - \frac{u}{\varepsilon}}{\sqrt{\varepsilon}} \right) = F_x
\]

for \( \varepsilon/\sqrt{\varepsilon} \geq u \geq \varepsilon/(\sqrt{\varepsilon} - p) \) for \( \varepsilon \)

(7)

\[
0 \geq \left( \frac{z}{\sqrt{\varepsilon}} + \left( \frac{z}{\sqrt{\varepsilon} + 1} - d \right) \left( \sqrt{\varepsilon} + 1 \right) \right) \left( \frac{u_0^2 \varepsilon^3 - \frac{u}{\varepsilon}}{\sqrt{\varepsilon}} \right) = F_x
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\]
Poisson approximation

If \( n \to \infty \) we get that
\[
\lim_{n \to \infty} \left( \frac{1}{2} \right)^{n/2} \frac{e^{-1/2}}{\sqrt{2\pi}} = \frac{1}{\sqrt{2\pi}}.
\]

Thus, to reach an acceptable accuracy, \( n \geq 3/(\varepsilon^2 - p) \).

Note that if \( \varepsilon < 0.1 \), we get that
\[
\frac{\varepsilon^2 - p}{\varepsilon} \approx \frac{\varepsilon}{(1 + \varepsilon)}.
\]

Therefore, we have the following theorem:

**Theorem 1.** For an arbitrary integer \( d \geq 0 \), we get the approximation (6) for \( \varepsilon \geq 0 \) \( d/\varepsilon \geq 0 \) that
\[
\frac{\varepsilon}{(1 + \varepsilon)} \approx \frac{\varepsilon}{(1 + \varepsilon)}.
\]

From this, we see that the condition (5) is satisfied when
\[
\frac{\varepsilon}{(1 + \varepsilon)} \approx \frac{\varepsilon}{(1 + \varepsilon)}.
\]

So, if \( \varepsilon \to \infty \) in Equation (5) does not work except
\[
\frac{\varepsilon}{(1 + \varepsilon)} \approx \frac{\varepsilon}{(1 + \varepsilon)}.
\]

We use the above quadratic approximation (6) to find, for example, for
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We use the above quadratic approximation (6) to find, for example, for
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\frac{\varepsilon}{(1 + \varepsilon)} \approx \frac{\varepsilon}{(1 + \varepsilon)}.
\]
focus on the range (3) for different values of \( p \) and \( \varepsilon \), keeping fixed. Of course, by symmetry, we can as well do
\[
\max (\varepsilon, d, u) \approx \frac{125}{12}
\]
if more interesting to look for
\[
\text{indentity.}
\]
By Remark 2, \( \chi^2 \) follows a Skellam distribution. We have that
\[
\begin{align*}
\frac{1}{2} & = \left( \frac{\chi^2}{\mu} \right) (\chi^2 + 1) + 1, \\
& \Downarrow
\end{align*}
\]
and its value, \( \varepsilon \), can be determined by identity (2), possibly using the approximation
\[
(0, d, u) \chi^2 - 1 = (0, d, u) \frac{\chi^2}{\mu}
\]
where \( \varepsilon \) is the modified Bessel function of the first kind of order \( \varepsilon \), for example, we get that \( \chi^2 \) follows the distribution of the difference of two independent Poisson random variables \( X - u \) and \( (d - 1)u \) and \( (d + 1)u \) with modified Bessel function of the second kind of order \( \varepsilon \), respectively. Therefore, \( (d + 1)u \sim X - u \) and \( (d - 1)u \sim X - u \).

\begin{table}[h]
\begin{center}
\begin{tabular}{cccccccc}
\hline
\( \varepsilon \) & Exact & Approximation & Exact & Approximation & Exact & Approximation & Exact & Approximation \\
\hline
0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 & 0.00 \\
0.05 & 0.05 & 0.05 & 0.05 & 0.05 & 0.05 & 0.05 & 0.05 & 0.05 \\
0.10 & 0.10 & 0.10 & 0.10 & 0.10 & 0.10 & 0.10 & 0.10 & 0.10 \\
\hline
\end{tabular}
\end{center}
\end{table}

Table 1: The values of \( f(n, \varepsilon) \) with \( \varepsilon = \varepsilon \) for \( \varepsilon = 0.00, 0.05, 0.10 \).
We observe that the shape of $f_d$ changes from concave up to concave down as $d$ increases. For $\varepsilon = 0$, the shape of $f_0$ is that of a distorted parabola opening up which is fairly flat about its vertex for all $n \geq 1$. The shape becomes concave down at $p = 1/2$ for $d \geq 1$.

For other values of $\varepsilon$, this change at $p = (1 + \varepsilon)/2$ happens at a higher value of $d$ depending on $n$ and $\varepsilon$, too. For example, with $\varepsilon = 0.3$, $(\partial^2/\partial p^2)f_d(12, 0.65, 0.3)$, $(\partial^2/\partial p^2)f_d(13, 0.65, 0.3)$, and $(\partial^2/\partial p^2)f_d(16, 0.65, 0.3)$ become negative at $d = 4$, $5$, and $6$, respectively.

As a consequence, typically, the maximum occurs at $p = 1$ when $d$ is small (cf. Figure 1(a), $f_0(12, p, 0.3)$). However, when it is not the case, the approximation methods of Sections 5 and 6 can hardly help. In fact, for $n = 12$, $\varepsilon = 0.05$, and $d = 1$ the optimum is found around $p = 0.8883$, and the probability appears to be sharply decreasing as $p$ increases from this value on (cf. Figure 1(b), $f_1(12, p, 0.05)$). On the other hand, as $d$ grows, it appears that the maximum occurs at $p = (1 + \varepsilon)/2$.

We prove only the following theorem.

**Theorem 7.1** The polynomial $f_0(n, p, 0)$ is concave up, and thus, it takes its minimum at $p = 1/2$. Its maximum is taken at 0 and 1.
With considerably more calculations, we obtain all coefficients which proves the convexity of the Mathematica package comparing the initial values of Equation (17) and the double sum is already guaranteed the convexity of \( T \). Lengyel (Δ1) δ

\[
\frac{1}{2} \left( \left( \frac{1 + u}{u - \frac{2}{1}} \right) \frac{\left( 1 - u \right) / \frac{2}{1} + 1}{1 - u} \right) = \frac{1}{2} \left( \left( \frac{1 + u}{u - \frac{2}{1}} \right) \frac{\left( 1 - u \right) / \frac{2}{1} + 1}{1 - u} \right)
\]

For instance, \( \Delta(1, 3) \), the simplifications, we obtain the recurrence relation \( \Delta(n, p, \Delta) = (0, u) \Delta \), where the initial condition follows immediately. With some calculations and

\[
(0, u - \frac{2}{1} + \Delta) \left( 1 / \frac{2}{1} \right) \sum_{u = 1}^{\Delta} \frac{z - u - \Delta}{u - \Delta} = \frac{1}{2} (0, u)^{\Delta}
\]

with the initial condition

\[
0 \geq p \text{ for } (u - \frac{2}{1} (u - p) \sum_{u = 1}^{\Delta} \frac{1 + p}{1} - (0, u)^{\Delta} = (1 + p, u)^{\Delta}
\]

For instance, we obtain the recurrence relation

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(0, u - \frac{2}{1} (u - p) \sum_{u = 1}^{\Delta} \frac{1 + p}{1} - (0, u)^{\Delta} = (1 + p, u)^{\Delta}
\]

with some calculations and

\[
(0, u - \Delta(1, 3) \left( \frac{1}{\frac{2}{1}} \right) \sum_{u = 1}^{\Delta} \frac{z - u - \Delta}{u - \Delta} = \frac{1}{2} (0, u)^{\Delta}
\]

The proof is complete after

\[
\text{Lemma: } \frac{z}{1 - d} = \Delta
\]
The results can be easily generalized for \( \lambda \). The constant term approximation (10) of Theorem 5.1 and numerical evidence suggest that \( f \) appeared in [9,4] with the pairs 

\[ \text{in this case the approximation (7) works even for small values of } \varepsilon. \]

It seems to matter as long as \( \varepsilon \) and the weaker player scores at conveniently switching the sign of \( \varepsilon/\lambda \). Since \( \lambda \) is small, in agreement with our findings regarding (B) and (C).

We can consider the absolute spread difference between \( X \) and \( \lambda \). Here we deal with the special case

\[ f/\lambda = b = d \]

The absolute spread difference for \( f/\lambda = b = d \)

moments of \( (x, p, \varepsilon)/\lambda \). The roots of \( f/\lambda = b = d \) appear approximately 

\[ \sqrt{\varepsilon} \lambda/\lambda \approx \varepsilon \lambda \]

Thus, \( \lambda \) is approximately of half-normal distribution, i.e. the distribution of the absolute

difference between \( X \) and \( \lambda \) as \( |X - \lambda| \) grows, leaving the original

moment of the distribution function unchanged at zero with the aid of moments of half-normal distribution.

We note that very recently the problem of finding the roots of 

\[ (\partial/\partial p)f(\partial/\partial x)f = 0 \]

for \( n \) at least \( 1 \), the author believes that

\[ f = \text{appears in } [9,4]. \]

We note that very recently the problem of finding

\[ \left( \begin{array}{c} f \\ -1 \end{array} \right) \left( \begin{array}{c} x \\ -1 \end{array} \right) \]

will be of considerable interest to prove the following theorem.

\[ \text{Theorem 7.} \]

\[ \lambda = \text{appears in } [9,4]. \]

We add this equation also implies that the function

\[ f/\lambda = b = d \]

We note that \( (\partial/\partial x)f \)

Theorem 7. \]

\[ \lambda = \text{appears in } [9,4]. \]

We add this equation also implies that the function

\[ f/\lambda = b = d \]

We note that \( (\partial/\partial x)f \)
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with the particular values

\[ m_0 = 1 - \left( \frac{2}{n} \right) \frac{n}{2} \]

\[ m_1 = \frac{n}{2} \frac{n}{2} \]

\[ m_2 = \frac{n}{2} \frac{n}{2} \]

\[ m_3 = \left( \frac{2}{n} \right) \frac{n}{2} \frac{n}{2} \frac{n}{2} \]

\[ m_4 = \left( \frac{3}{n} \right) \frac{n}{4} - \frac{1}{2} \]

\[ m_5 = \left( \frac{2}{n} \right) \frac{n}{2} \frac{n}{2} \frac{n}{2} \frac{n}{2} \]

\[ m_6 = \left( \frac{15}{n} \right) \frac{n}{8} - \frac{15}{n} + \frac{4}{n} \]

Of course, \( E(\left| X - Y \right|^2) = \text{var}(X - Y) = \frac{n}{2} \)

We also observe that

\[ \frac{1}{u} \left( \frac{u}{u} \right) \frac{u}{u} = \frac{v}{v} \]

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The closed form for \( m_1 \) was originally suggested by John Essam and derived in [7].

We note that Theorem 8.1 is in agreement with the moments of the corresponding half-normal distribution in an asymptotic sense.

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References


