PRODUCT OF FACTORIALS IN THE SEQUENCE \( \{g_n\} \)

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**Abstract.** Let \( \{g_n\}_{n \geq 0} \) defined as \( g_n = g_{n-1} + g_{n-2} \) with \( g_1 = 1 \) and \( g_2 = a \) (\( a \in \mathbb{Z}^+ \)). We characterize \( 2 \)-adic valuation of the sequence \( \{g_n\} \) for \( a \equiv 3, 4, 5, 6 \pmod{8} \). Afterwards, we solve the equation

\[ g_n = m_1! m_2! \cdots m_k! \]

completely.

1. **Introductions**

Several mathematicians are interested in finding factorials in special sequences as Fibonacci, Lucas etc. Luca [2] showed that the terms of \( F_3, F_6, F_{12}, L_0 \) and \( L_3 \) can be written as the products of the factorials where \( F_n \) and \( L_n \) are \( n \)-th Fibonacci and Lucas numbers, respectively. Moreover, the largest product of distinct Fibonacci numbers which is a product of factorials was shown in [3] by Luca. Later, Grossman and Luca [4] proved that the equation

\[ F_n = m_1! + m_2! + \cdots + m_k! \]

has finitely many positive integers \( n \) for fixed \( k \) integer. The case \( k \leq 2 \) has been determined. The case \( k = 3 \) was solved by Bollman, Hernandez and Luca.

The \( p \)-adic order, \( \nu_p(r) \), of \( r \) is the exponent of the highest power of a prime \( p \) which divides \( r \). Recently, Marques and Lengyel [5] characterized \( 2 \)-adic valuation of \( T_n \) and showed that \( T_n \) is factorial when \( n = 1, 2, 3 \) and 7. For other details about the special sequences, we refer the papers [7] and [8].

Let \( a \in \mathbb{Z}^+ \). For \( n \geq 3 \), define the sequence \( \{g_n\} \) as

\[ g_n = g_{n-1} + g_{n-2} \]

with \( g_1 = 1 \) and \( g_2 = a \). We get Fibonacci and Lucas sequence if taking \( a = 1 \) and \( a = 3 \), respectively. In this paper we characterize

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**2010 Mathematics Subject Classification.** 11B39, 11D72.

**Key words and phrases.** Factorials, recurrence sequences, diophantine equations.
2–adic order of \( g_n \) for \( a \equiv 3, 4, 5, 6 \pmod{8} \) and solve the equation \( g_n = \prod_{j=1}^{k} m_j \).

Our theorems are following,

**Theorem 1.** For \( n \geq 1 \), we have
\[
\nu_2(g_n) = \begin{cases} 
0 & \text{if } n \equiv 1, 2 \pmod{3} \\
\nu_2(a-1) & \text{if } n \equiv 0 \pmod{6} \\
\nu_2(a+1) & \text{if } n \equiv 3 \pmod{6}
\end{cases}
\]
for \( a \equiv 3, 5 \pmod{8} \).

If \( a \equiv 4, 6 \pmod{8} \), then
\[
\nu_2(g_n) = \begin{cases} 
0 & \text{if } n \equiv 0, 1 \pmod{3} \\
\nu_2(a) & \text{if } n \equiv 2 \pmod{6} \\
\nu_2(a-2) & \text{if } n \equiv 5 \pmod{6}
\end{cases}
\]

**Remark 1.** As seen above theorem, \( \nu_2(g_n) \leq 2 \) follows for \( n \geq 1 \).

**Theorem 2.** Assume that \( m_i \geq 2 \) (\( 1 \leq i \leq k \)). Then the solutions of the equation
\[
(1.1) \quad g_n = \prod_{j=1}^{k} m_j!
\]
are given as follows:

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Before proceeding further, some considerations will be needed for the convenience of the reader.

**Lemma 1.** Let \( m \) and \( n \) be positive integers and \( p \) is a prime number. If \( \nu_p(n) \neq \nu_p(m) \), then
\[
\nu_p(m + n) = \inf \{ \nu_p(n), \nu_p(m) \}
\]
holds.

**Lemma 2.** For any integer \( k \geq 1 \) and \( p \) prime, we have
\[
\frac{k}{p-1} - \left\lfloor \frac{\log k}{\log p} \right\rfloor - 1 \leq \nu_p(k!) \leq \frac{k-1}{p-1}
\]
where \( \lfloor x \rfloor \) denotes the largest integer less than or equal to \( x \).

**Proof.** We refer Lemma 2.4 in the paper of Marques [6]. \(\square\)
Lemma 3. For $n, k$ and $s$ are positive integers, we get

$$g_{rn+s} = L_r g_{r(n-1)+s} - (-1)^r g_{r(n-2)+s}$$

where $L_r$ is $r$th Lucas number and $0 \leq s \leq r - 1$.

Proof. It can be proven by the binet formula of the sequence $\{g_n\}$. □

2. Proof of Theorem 1

We will prove only the case $a \equiv 3 \pmod{8}$. The other cases can be proven by using the similar way. In order to show $\nu_2(g_n) = 0$, we need to prove that $g_n \equiv 1 \pmod{2}$. To avoid unnecessary repetitions we shall prove only that case $n \equiv 1 \pmod{3}$. For that, we shall proceed by induction on $n$. The base case $n = 1, g_1 = 1$. We may suppose that $g_{3n-2} \equiv 1 \pmod{2}$ and $g_{3n-5} \equiv 1 \pmod{2}$. By Lemma 3, we deduce that

$$g_{3n+1} = 4g_{3n-2} + g_{3n-5}.$$ 

After taking modulo 2 of both sides, then

$$g_{3n+1} \equiv 4 \cdot 1 + 1 \pmod{2}$$
$$\equiv 1 \pmod{2}$$

follows as claimed.

Now assume that $n \equiv 0 \pmod{6}$. Now the base case is $n = 6$. Since $g_6 = 5a + 3$ and $a \equiv 3 \pmod{8}$, then

$$\nu_2(g_6) = \nu_2(5a + 3)$$
$$= \nu_2(5(8k + 3) + 3)$$
$$= \nu_2(40k + 18)$$

for some $k \in \mathbb{Z}^+$. As $1 = \nu_2(40k + 10) \neq \nu_2(8) = 3$, then we obtain that $\nu_2(g_6) = \nu_2(40k + 10)$ by Lemma 1. It yields that $\nu_2(g_6) = \nu_2(a - 1)$ as claimed. As $\nu_2(a - 1) = 1$ for $a \equiv 3 \pmod{8}$, we will show $g_n \equiv 2 \pmod{4}$ for $n \equiv 0 \pmod{6}$. Assume that $g_{6(n-1)} \equiv 2 \pmod{4}$ and $g_{6(n-2)} \equiv 2 \pmod{4}$. By Lemma 3, we have

$$g_{6n} = L_6 g_{6(n-1)} - g_{6(n-2)}$$
$$= 18g_{6(n-1)} - g_{6(n-2)}.$$ 

Then $g_{6n} \equiv 2 \pmod{4}$ follows which gives that $\nu_2(g_{6n}) = \nu_2(a - 1) = 1$. Since the case $n \equiv 3 \pmod{6}$ can be proven similarly, we omit this case. Therefore, we prove the Theorem 1.
3. Proof of Theorem 2

Assume that \( k \geq 3 \). Then we arrive at a contradiction after taking 2−adic valuation of both sides of the equation (1.1) since \( \nu_2 \left( \prod_{j=1}^{k} m_j! \right) \geq 3 \) and \( \nu_2 (g_n) \leq 2 \). So, \( k = 1 \) and \( k = 2 \) follow. The possible solutions are given in Theorem 2.

4. Open Question

In this paper, we characterize 2−adic order \( g_n \) for \( a \equiv 3, 4, 5, 6 \) (mod 8). What is 2−adic valuation of the \( g_n \) for \( a \equiv 0, 1, 2, 7 \) (mod 8)? We leave this problem as a question for reader.

References


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