Congruences for Stirling Numbers of the Second Kind Modulo 5

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Received 10 February 2013
Accepted 28 July 2013

Communicated by K.P. Shum

AMS Mathematics Subject Classification(2000): 11B73, 11A07

Abstract. Let $n$ and $k$ be positive integers and $S(n, k)$ be the Stirling numbers of the second kind. In this paper, the author establishes the congruences for $S(n, k)$ modulo 5 in terms of binomial coefficients.

Keywords: Stirling number of the second kind; Congruence; Binomial coefficient

1. Introduction

The Stirling numbers are common topics in number theory and combinatorics. Let $\mathbb{N}$ denote the set of natural numbers. The Stirling numbers of the first kind, denoted by $s(n, k)$ (with a lower-case “$s$”), count the number of permutations of $n$ elements with $k$ disjoint cycles. The Stirling numbers of the second kind $S(n, k)$ (with a capital “$S$”) is defined for $n \in \mathbb{N}$ and positive integer $k \leq n$ as the number of ways to partition a set of $n$ elements into exactly $k$ non-empty subsets. One can characterize the Stirling numbers of the first and the second kind by

\[
\max\{j, k\} \sum_{l=0}^{\max\{j, k\}} s(l, j)S(k, l) = \delta_{jk} \quad \text{and} \quad \max\{j, k\} \sum_{l=0}^{\max\{j, k\}} S(l, j)s(k, l) = \delta_{jk},
\]

where $\delta_{jk}$ is the Kronecker delta.
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Divisibility is an important and interesting topic in number theory. See [5], [6] and [12] for some interesting results on this topic. For example, the elements of an arbitrary gcd-closed set of the form $pq^n$ were considered and studied in [6]. Divisibility properties of Stirling numbers have been studied from a number of different perspectives. Given a prime $p$ and a positive integer $m$, there exist unique integers $a$ and $m$, with $p \nmid a$ and $n \geq 0$, such that $m = ap^n$. The number $n$ is called the $p$-adic valuation of $m$, denoted by $n = v_p(m)$. The numbers $\min\{v_p(k!S(n,k)) : m \leq k \leq n\}$ are important in algebraic topology, see [2]. Lengyel [9] studied the 2-adic valuations of $S(n,k)$ and conjectured, proved by Wannemacker [11], that $v_2(S(2^n,k)) = s_2(k) - 1$, where $s_2(k)$ means the base 2 digital sum of $k$. Lengyel [10] showed that if $1 \leq k \leq 2^n$, then $v_2(S(c2^n,k)) = s_2(k) - 1$ for any positive integer $c$. Amdeberhan, Manna and Moll [1] suggested that $v_2(S(2^n + 1,k + 1)) = s_2(k) - 1$, which confirmed by Hong, Zhao and Zhao [7].

Congruence is another central topic in the field of Stirling numbers of the second kind. It is known that for each fixed $k$, the sequence $\{S(n,k), n \geq k\}$ is periodic modulo prime powers. The length of this period has been studied by Carlitz [3] and Kwong [8]. Chan and Manna [4] characterized $S(n,k)$ modulo prime powers in terms of binomial coefficients when $k$ is a multiple modulus.

In this paper, we establishes the congruences for $S(n,k)$ modulo 5 in terms of binomial coefficients. For any real number $x$, we denote $\lfloor x \rfloor$ by the biggest integer no more than $x$. In fact, we have the following main results.

**Theorem 1.1.** Let $n$ and $a$ be positive integers with $n \geq 5a$. Then the following congruences are true modulo 5:

(i) $S(n,5a) \equiv \begin{cases} \frac{n-1}{a} & \text{if } n \equiv a \pmod{4}, \\ 0 & \text{otherwise.} \end{cases}$

(ii) $S(n,5a + 1) \equiv \frac{2}{a} \cdot \left( \frac{n-1}{a} \right).$

(iii) $S(n,5a + 2) \equiv \begin{cases} 2 \cdot \left( \frac{n-1}{a} \right) & \text{if } n \equiv a - 1 \pmod{4}, \\ 3 \cdot \left( \frac{n-1}{a} \right) & \text{if } n \equiv a - 2 \pmod{4}, \\ 0 & \text{otherwise.} \end{cases}$

(iv) $S(n,5a + 3) \equiv \begin{cases} \frac{n-1}{a} & \text{if } n \equiv a \pmod{4}, \\ 0 & \text{if } n \equiv a - 1 \pmod{4}, \end{cases}$

(v) $S(n,5a + 4) \equiv \begin{cases} \frac{n-1}{a} & \text{if } n \equiv a \pmod{4}, \\ 0 & \text{otherwise.} \end{cases}$

In the next section, we will show Theorem 1.1.
2. Proof of Theorem 1.1

At first, we will use the elementary property of $S(n, k)$ as follows. For each fixed $k$, $S(n, k)$ has generating function

$$
\sum_{n=0}^{\infty} S(n, k)x^n = \prod_{i=1}^{k} \frac{x}{1 - ix}.
$$

(i). By (1), we have

$$
\sum_{n=0}^{\infty} S(n, 5a)x^n = \prod_{i=1}^{5a} \frac{x}{1 - ix} = \left( \prod_{i=1}^{5} \frac{x}{1 - ix} \right)^a
\equiv x^{5a} \left( \frac{1}{1 - x^4} \right)^a = x^{5a} (1 - x^4)^{-a}
= x^{5a} \sum_{j=0}^{\infty} \binom{-a}{j} (-x^4)^j
= \sum_{j=0}^{\infty} \binom{a - 1 + j}{j} x^{4j + 5a} \pmod{5}. \tag{2}
$$

Collecting powers and reindexing in (2), we derive that if $n \equiv a \pmod{4}$, then

$$
S(n, 5a) \equiv \left( \frac{a - 1 + \frac{n-5a}{5a}}{\frac{n-5a}{5a}} \right) \equiv \left( \frac{n-a-4}{a-1} \right) \pmod{5},
$$

and if $n \not\equiv a \pmod{4}$, then $S(n, 5a) \equiv 0 \pmod{5}$. So part (i) is proved.

(ii). From (1), we can deduce that

$$
\sum_{n=0}^{\infty} S(n, 5a + 1)x^n = \prod_{i=1}^{5a+1} \frac{x}{1 - ix} \equiv \left( \prod_{i=1}^{5a} \frac{x}{1 - ix} \right) \cdot \frac{x}{1 - x}
\equiv \left( \sum_{m=0}^{\infty} S(m, 5a)x^m \right) \left( \sum_{r=1}^{\infty} x^r \right) = \sum_{n=0}^{\infty} \sum_{t=0}^{n-1} S(t, 5a)x^n \pmod{5}. \tag{3}
$$

Then by (3) and the definition of Stirling numbers of the second kind, we get

$$
S(n, 5a + 1) \equiv \sum_{t=0}^{n-1} S(t, 5a) = \sum_{t=5a}^{n-1} S(t, 5a) \pmod{5}. \tag{4}
$$

It then follows from (4) and part (i) that

$$
S(n, 5a + 1) \equiv \sum_{\substack{5a \leq s \leq n-1 \\ j=0 \text{ mod } 4}} \binom{\frac{5a-1}{a}}{a} \equiv \sum_{\substack{5a \leq s \leq n-1 \\ j=0 \text{ mod } 4}} \binom{\frac{j+5a}{a}}{a} \equiv \sum_{s=\frac{a-1}{a}}^{\frac{n-1-5a}{a}} \binom{s-1}{a} \equiv \sum_{j=0}^{\frac{n-1-5a}{a}} \binom{j+a-1}{a} \pmod{5}. \tag{5}
$$
Thus by (5) and applying the identity
\[
\sum_{j=0}^{c} \binom{b+j}{b} = \binom{b+1+c}{b+1}
\]
with \( b = a - 1 \) and \( c = \lfloor \frac{n-a-2}{4} \rfloor \), we obtain
\[
S(n, 5a + 1) \equiv \left( \frac{n-a-1}{a} \right) \pmod{5}
\]
as required. Part (ii) is true.

(iii). By (1), we get that
\[
\sum_{n=0}^{\infty} S(n, 5a + 2)x^n = (\prod_{k=1}^{5a+5} \frac{x}{1-kx}) \frac{(1-3x)(1-4x)(1-5x)}{x^3}
\]
\[
\equiv \left( \sum_{m=0}^{\infty} S(m, 5(a+1))x^m \frac{1+3x+2x^2}{x^3} \right)
\]
\[
= \sum_{n=0}^{\infty} S(n+3, 5(a+1))x^n + 3 \sum_{n=0}^{\infty} S(n+2, 5(a+1))x^n
\]
\[
+ 2 \sum_{n=0}^{\infty} S(n+1, 5(a+1))x^n \pmod{5}.
\]
It follows that
\[
S(n, 5a + 2) \equiv S(n+3, 5(a+1)) + 3S(n+2, 5(a+1))
\]
\[
+ 2S(n+1, 5(a+1)) \pmod{5}.
\] (6)

On the other hand, by part (i) we derive that
\[
S(n+3, 5(a+1)) \equiv \begin{cases} 
\frac{(n+3-a-2)}{a} & \text{if } n+3 \equiv a+1 \pmod{4} \\
0 & \text{otherwise}
\end{cases} \pmod{5},
\] (7)
\[
S(n+2, 5(a+1)) \equiv \begin{cases} 
\frac{(n+2-a-2)}{a} & \text{if } n+2 \equiv a+1 \pmod{4} \\
0 & \text{otherwise}
\end{cases} \pmod{5},
\] (8)
\[
S(n+1, 5(a+1)) \equiv \begin{cases} 
\frac{(n+1-a-2)}{a} & \text{if } n+1 \equiv a+1 \pmod{4} \\
0 & \text{otherwise}
\end{cases} \pmod{5}.
\] (9)

It then follows from (6), (7), (8) and (9) that
\[
S(n, 5a + 2) \equiv \begin{cases} 
2\left( \frac{n-a-2}{a} \right) & \text{if } n \equiv a \pmod{4}, \\
3\left( \frac{n-a-2}{a} \right) & \text{if } n \equiv a - 1 \pmod{4}, \\
0 & \text{if } n \equiv a - 2 \pmod{4}
\end{cases} \pmod{5}
\]
as desired. Part (iii) is proved.

(iv). From (1), we derive that

\[ \sum_{n=0}^{\infty} S(n, 5a + 3)x^n = \left( \prod_{i=1}^{5a+5} \frac{x}{1 - ix} \right) \frac{(1 - 4x)(1 - 5x)}{x^2} \]

\[ \equiv \left( \sum_{m=0}^{\infty} S(m, 5(a + 1))x^m \right) \frac{1 + x}{x^2} \]

\[ = \sum_{n=0}^{\infty} \left( S(n + 2, 5(a + 1)) + S(n + 1, 5(a + 1)) \right) x^n \pmod{5}. \] (10)

Thus by (10) we know that

\[ S(n, 5a + 3) \equiv S(n + 2, 5(a + 1)) + S(n + 1, 5(a + 1)) \pmod{5}. \] (11)

It then follows from (8), (9) and (11) that if \( n \equiv a + 3 \pmod{4} \), then

\[ S(n, 5a + 3) \equiv S(n + 2, 5(a + 1)) \equiv \left( \frac{n-a-3}{a} \right) \pmod{5}; \]

if \( n \equiv a + 2 \pmod{4} \), then

\[ S(n, 5a + 3) \equiv S(n + 1, 5(a + 1)) \equiv \left( \frac{n-a-4}{a} \right) \pmod{5}; \]

if ether \( n \equiv a + 1 \pmod{4} \) or \( n \equiv a \pmod{4} \), then \( S(n, 5a + 3) \equiv 0 \pmod{5} \). This implies that part (iv) is true.

(v). Using (1), we can obtain that

\[ \sum_{n=0}^{\infty} S(n, 5a + 4)x^n = \left( \prod_{i=1}^{5a+5} \frac{x}{1 - ix} \right) \frac{(1 - 5x)}{x} \]

\[ \equiv \left( \sum_{m=0}^{\infty} S(m, 5(a + 1))x^m \right) x^{-1} = \sum_{n=0}^{\infty} S(n + 1, 5(a + 1))x^n \pmod{5}. \] (12)

It then follows from (9) and (12) that

\[ S(n, 5a + 4) \equiv S(n + 1, 5(a + 1)) \equiv \begin{cases} \left( \frac{n-a-4}{a} \right) & \text{if } n \equiv a \pmod{4}, \\ 0, & \text{otherwise} \end{cases} \pmod{5} \]

as required. Part (v) is proved.

The proof of Theorem 1.1 is complete.

References


