On Diophantine Equations Related to Narayana’s Cows Sequence and Double Factorials or Repdigits

Yanjiao Ji¹, Peng Yang¹,⁎ and Tianxin Cai²

¹ School of Science, University of Science and Technology Liaoning, Anshan 114051, China
² Department of Mathematics, Zhejiang University, Hangzhou 310027, China
⁎ Correspondence: 32012880106@ustl.edu.cn

Abstract: In this paper, we determine all the Narayana’s cows numbers that are factorials or double factorials. We also show that 88 is the only repdigit (i.e., a class of numbers that has reflectional symmetry across a vertical axis) that can be written as the product of consecutive Narayana’s cows numbers.

Keywords: factorial; repdigit; Narayana’s cows sequence; Diophantine equation

1. Introduction

In 1356, Indian mathematician Narayana Pandita proposed the problem of a herd of cows and calves in his famous book titled Ganita Kaumudi [1]. It is a problem similar to Fibonacci’s rabbit problem. One can see that the number of cows in each year forms a sequence with the first few terms:

1, 1, 1, 2, 3, 4, 6, 9, 13, 19, 28, 41, 60, 88, 129, 189, 277, …

This sequence is named Narayana’s cows sequence. It is also called the Fibonacci–Narayana sequence or Narayana sequence. However, there is another sequence that is also named the Narayana sequence (see [2]). The Narayana’s cows sequence can be written as the following recurrence:

Gₙ = Gₙ₋₁ + Gₙ₋₃

for n ≥ 3 with initial values G₀ = G₁ = G₂ = 1.

In the literature, there are several results dealing with Diophantine equations involving factorials, repdigits, and recurrence sequences. In 1999, Luca [3] proved that F₁₂ = 2!² · 3!² = 3! · 4! and L₃ = (2!)² are the largest Fibonacci and Lucas numbers that can be represented as the products of factorials, respectively. In 2006, Luca and Stănică [4] found all products of Fibonacci numbers that are products of factorials. Those results can be proven by applying the primitive divisor theorem. Meanwhile, a high-order recurrence version of the primitive divisor theorem seems to be out of reach. By characterizing the 2-adic valuation of tribonacci numbers, Marques and Lengyel [5] determined all the factorials in a tribonacci sequence. Using the same method, Irmak [6] identified the factorials in Perrin or Padovan sequences, and Guadalupe [7] found all the factorials in Narayana’s cows numbers.

A repdigit is a positive integer with only one distinct digit in its decimal expansion. It has the form a(10ᵐ − 1)/9 for some m ≥ 1 and 1 ≤ a ≤ 9. In 2000, Luca [8] showed that the largest repdigits in Fibonacci and Lucas sequences are F₁₀ = 55 and L₅ = 11. Since then, this result has been generalized and extended in various directions. For example, Faye and Luca [9] proved that P₃ = 5 and Q₂ = 6 are the largest repdigits in Pell and Pell–Lucas sequences, respectively. Bravo et al. [10] obtained all base b repdigits that are the sum of two Narayana numbers. Considering the consecutive product of the recurrence sequence,
Marques and Togbé [11] showed that the product of consecutive Fibonacci numbers can never be a repdigit greater than 10. Irmak and Togbé [12] verified that the largest repdigit appearing as the product of consecutive Lucas numbers is 77. Rayaguru and Panda [13] studied repdigits as products of consecutive balancing and Lucas-balancing numbers. Bravo et al. [14] proved that 44 is the largest repdigit in the product of consecutive tribonacci numbers. Recently, Rihane and Togbé [15] dealt with repdigits that can be written as the products of consecutive Padovan or/and Perrin numbers.

Motivated by the results of [3–6, 8–15], it is natural to ask what will happen if we consider Narayana’s cows numbers. We denote the double factorial \( n!! \) as the product of the natural numbers less than or equal to \( n \) that have the same parity as \( n \). In this paper, we investigate double factorials and repdigits in Narayana’s cows sequence. We mainly solve the Diophantine equation \( G_n = m!! \) and study repdigits that can be written as the product of consecutive Narayana’s cows numbers. More precisely, we prove the following results.

**Theorem 1.** The only solutions of the Diophantine equation

\[ G_n = m!! \]

in positive \( n, m \) are

\[ (n, m) \in \{(1, 1), (2, 1), (3, 2), (4, 3)\}. \]

**Theorem 2.** The only solution of the Diophantine equation

\[ G_n \cdots G_{n+\ell-1} = a \left( \frac{10^m - 1}{9} \right) \]  

in positive integers \( n, \ell, m, a \), with \( 1 \leq a \leq 9 \) and \( m \geq 2 \) is

\[ (n, \ell, m, a) = (13, 1, 2, 8), \quad \text{i.e.,} \quad G_{13} = 88. \]

From the above theorem, we have the following corollary.

**Corollary 1** (Bravo et al. [10]). The only repdigit in the Narayana’s cows sequence is \( G_{13} = 88 \).

### 2. Auxiliary Results

#### 2.1. Narayana’s Cows Sequences and Its \( p \)-adic Valuations

There are few properties of Narayana’s cows sequences \( \{G_n\}_{n \geq 0} \) that are known. The characteristic equation of \( \{G_n\}_{n \geq 0} \) is \( x^3 - x^2 - 1 = 0 \). It has one real root \( \alpha \) and two complex roots \( \beta \) and \( \gamma = \bar{\beta} \). More precisely,

\[
\alpha = \frac{1}{3} \left( \sqrt[3]{\frac{1}{2} (29 - 3\sqrt{93})} + \sqrt[3]{\frac{1}{2} (3\sqrt{93} + 29)} + 1 \right), \\
\beta = \frac{1}{3} - \frac{1}{6} (1 - i\sqrt{3}) \sqrt[3]{\frac{1}{2} (29 - 3\sqrt{93})} - \frac{1}{6} (1 + i\sqrt{3}) \sqrt[3]{\frac{1}{2} (3\sqrt{93} + 29)}, \\
\gamma = \frac{1}{3} - \frac{1}{6} (1 + i\sqrt{3}) \sqrt[3]{\frac{1}{2} (29 - 3\sqrt{93})} - \frac{1}{6} (1 - i\sqrt{3}) \sqrt[3]{\frac{1}{2} (3\sqrt{93} + 29)}. 
\]

For all \( n \geq 0 \), the Narayana’s cows sequence satisfying the following “Binet-like” formula (see [16])

\[ G_n = c_\alpha \alpha^n + c_\beta \beta^n + c_\gamma \gamma^n, \]

where

\[
c_\alpha = \frac{(1 - \beta)(1 - \gamma)}{(\alpha - \beta)(\alpha - \gamma)}, \quad c_\beta = \frac{(1 - \alpha)(1 - \gamma)}{(\beta - \alpha)(\beta - \gamma)}, \quad \text{and} \quad c_\gamma = \frac{(1 - \alpha)(1 - \beta)}{(\gamma - \alpha)(\gamma - \beta)}. 
\]
The coefficient \( c_\alpha \) can be simplified as
\[
c_\alpha = \frac{1 - (\beta + \gamma) + \beta \gamma}{\alpha^2 - \alpha(\beta + \gamma) + \beta \gamma} = \frac{1 - (1 - \alpha) + \frac{1}{\alpha}}{\alpha^2 - \alpha(1 - \alpha) + \frac{1}{\alpha}} = \frac{\alpha^2 + 1}{2\alpha^3 - \alpha^2 + 1}.
\]

Then, by symmetry, we also have
\[
c_\beta = \frac{\beta^2 + 1}{2\beta^3 - \beta^2 + 1}, \quad \text{and} \quad c_\gamma = \frac{\gamma^2 + 1}{2\gamma^3 - \gamma^2 + 1}.
\]

It is easy to see that \( \alpha \in (1.46, 1.47) \), \( |\beta| = |\gamma| \in (0.82, 0.83) \), \( c_\alpha \in (0.61, 0.62) \), and \( |c_\beta| = |c_\gamma| \in (0.57, 0.58) \).

From the facts that \( \beta = \alpha^{-1/2}e^{i\theta} \) and \( \gamma = \alpha^{-1/2}e^{-i\theta} \), for some \( \theta \in (0, 2\pi) \), we can find that
\[
G_n = c_\alpha \alpha^n + e_n, \quad \text{with} \quad |e_n| < \frac{1}{\alpha^{n/2}}, \quad \text{for all} \ n \geq 1.
\]

By the induction method, it is not difficult to prove the following lemma.

**Lemma 1** ([16]). For all \( n \geq 2 \), we have
\[
\alpha^{n-2} \leq G_n \leq \alpha^{n-1}.
\]

**Lemma 2** (Guadalupe [7]). For all non-negative integers \( n, w \) with \( w \geq 4 \), we have
\[
G_{n+w} = G_{n-3}G_n + G_{n-4}G_{n+1} + G_{n-2}G_{n+2}.
\]

It is not easy to characterize the 2-adic order of Narayana’s cows numbers. However, to prove Theorem 2, we only need to give its lower bound. Moreover, it is easy to verify the following lemma by induction.

**Lemma 3.** For \( n \geq 1 \), we have
\[
\begin{align*}
v_2(G_n) &\geq 1, \quad \text{if} \ n \equiv 3, 6 \pmod{7}, \\v_2(G_n) &\geq 2, \quad \text{if} \ n \equiv 5 \pmod{7}.
\end{align*}
\]

**Proof.**
(i) For \( n \equiv 3(\text{mod} \ 7) \), let \( n = 7k + 3 \) and proceed by induction on \( k \). If \( k = 0 \), then \( n = 3 \) and \( G_3 = 2 \). If \( k = 1 \), then \( n = 10 \) and \( G_{10} = 28 \). Suppose \( k = m \) and \( 2 \mid G_{7m+3} \).

Then,
\[
G_{7(m+1)+3} = G_{7(m+3)+7} = G_{7-3}G_{7m+3} + G_{7-4}G_{7m+3+1} + G_{7-2}G_{7m+3+2} = G_4G_{7m+3} + G_3G_{7m+4} + G_5G_{7m+5}
\]

Since \( 2 \mid G_{7m+3}, 2 \mid G_3, \) and \( 2 \mid G_5 = 4 \), we have \( G_{7(m+1)+3} \equiv 0(\text{mod} \ 2) \).

(ii) For \( n \equiv 6(\text{mod} \ 7) \), let \( n = 7k + 6 \) and proceed by induction on \( k \). If \( k = 0 \), then \( n = 6 \) and \( G_6 = 6 \). If \( k = 1 \), then \( n = 13 \) and \( G_{13} = 88 \). Suppose \( k = m \) and \( 2 \mid G_{7m+6} \).

Then,
\[
G_{7(m+1)+6} = G_{7(m+6)+7} = G_{7-3}G_{7m+6} + G_{7-4}G_{7m+6+1} + G_{7-2}G_{7m+6+2} = G_4G_{7m+6} + G_3G_{7m+7} + G_5G_{7m+8}
\]

Since \( 2 \mid G_{7m+6}, 2 \mid G_3, \) and \( 2 \mid G_5 \), we have \( G_{7(m+1)+6} \equiv 0(\text{mod} \ 2) \).
(iii) For \( n \equiv 5(\text{mod } 7) \), let \( n = 7k + 5 \) and proceed by induction on \( k \). If \( k = 0 \), then \( n = 5 \) and \( G_5 = 4 \). If \( k = 1 \), then \( n = 12 \) and \( G_{12} = 60 \). Suppose \( k = m \) and \( 4 \mid G_{7m+5} \). Then,

\[
G_{7(m+1)+5} = G_{7m+5} + 7 = G_7 G_{7m+5} + G_7 - 4 G_{7m+5+1} + G_7 - 2 G_{7m+5+2} = G_4 G_{7m+5} + G_3 G_{7m+6} + G_5 G_{7m+7}
\]

According to (ii), we have \( 2 \mid G_{7m+6} \) and \( 4 \mid G_3 G_{7m+6} \). Thus, by \( 4 \mid G_{7m+5} \) and \( 4 \mid G_5 \), we have \( G_{7(m+1)+5} \equiv 0(\text{mod } 4) \).

Therefore, we obtain the following lower bound.

**Corollary 2.** For \( n \geq 1 \) and \( \ell \geq 7 \),

\[
v_2(G_n G_{n+1} \cdots G_{n+(\ell-1)}) \geq 4.
\]

To prove Theorem 1, we need the following lemmas.

**Lemma 4** (Guadalupe [7]). For \( n \geq 1 \), we have

\[
v_3(G_n) = \begin{cases} 
0, & \text{if } n \equiv 0, 1, 2, 3, 5 \pmod{8}, \\
1, & \text{if } n \equiv 4, 6, 12, 14 \pmod{24}, \\
2, & \text{if } n \equiv 7 \pmod{24}, \\
v_3(n + 9) + 2, & \text{if } n \equiv 15 \pmod{24}, \\
v_3(n + 4) + 1, & \text{if } n \equiv 20 \pmod{24}, \\
v_3(n + 2) + 1, & \text{if } n \equiv 22 \pmod{24}, \\
v_3(n + 1) + 2, & \text{if } n \equiv 23 \pmod{24},
\end{cases}
\]

(6)

where \( v_p(r) \) is the exponent of prime \( p \) in the factorization of \( r \).

**Lemma 5** (Grossman, Luca [17]). For any prime \( p \) and positive integer \( n \geq p \), we have

\[
v_p(n!) > \frac{n}{2p}.
\]

From the above lemma, we have a similar inequality for double factorials.

**Lemma 6.** For any odd prime \( p \) and positive integer \( n \geq p \), we have

\[
v_p(n!!) > \frac{n}{4p}.
\]

**Proof.** If \( n \) is even, then

\[
v_p(n!!) = v_p\left(2^n \left(\frac{n}{2}\right)!ight) = v_p\left(\left(\frac{n}{2}\right)!ight) > \frac{n}{4p}.
\]

If \( n \) is odd, then

\[
v_p(n!!) \geq v_p((n + 1)!!).
\]

Thus,

\[
v_p(n!!) \geq \frac{v_p(n!!)}{2} > \frac{n}{4p}.
\]

\( \square \)
2.2. Linear Forms in Logarithms

For any non-zero algebraic number $\gamma$ of degree $d$ over $\mathbb{Q}$, whose minimal polynomial over $\mathbb{Z}$ is $\prod_{j=1}^{d} (X - \gamma^{(j)})$, we denote by

$$h(\gamma) = \frac{1}{d} \left( \log |a| + \sum_{j=1}^{d} \log \max(1, \left| \gamma^{(j)} \right|) \right)$$

the usual absolute logarithmic height of $\gamma$.

To prove Theorem 2, we use lower bounds for linear forms in logarithms to bound the subscript $n$ appearing in Equation (1). We quote the following result.

Lemma 7 (Bugeaud et al. [18], Matveev [19]). Let $\gamma_1, \ldots, \gamma_s$ be real algebraic numbers and let $b_1, \ldots, b_s$ be the non-zero rational integer numbers. Let $D$ be the degree of the number field $\mathbb{Q}(\gamma_1, \ldots, \gamma_s)$ over $\mathbb{Q}$ and let $A_j$ be a positive real number satisfying

$$A_j = \max\{ Dh(\gamma), |\log \gamma|, 0.16 \} \text{ for } j = 1, \ldots, s.$$

Assume that $B \geq \max\{|b_1|, \ldots, |b_s|\}$.

If $\gamma_1^{b_1} \cdots \gamma_s^{b_s} \neq 1$, then

$$|\gamma_1^{b_1} \cdots \gamma_s^{b_s} - 1| \geq \exp(-C(s, D)(1 + \log B) A_1 \cdots A_s),$$

where $C(s, D) := 1.4 \cdot 30^{s+3} \cdot s^{4.5} \cdot D^2(1 + \log D)$.

The next step is to reduce the bound of $n$, which is generally too large. To this end, we present a variant of the reduction method of Baker and Davenport, which was introduced by de Weger [20].

Let

$$\Lambda = \beta + x_1 \vartheta_1 + x_2 \vartheta_2,$$

where $\vartheta_1, \vartheta_2 \in \mathbb{R}$ are given, and $x_1, x_2 \in \mathbb{Z}$ are unknowns. Set $X = \max\{|x_1|, |x_2|\}$. Let $X_0, Y$ be positive numbers. Assume that

$$|\Lambda| < c \cdot \exp(-\delta \cdot Y),$$

and

$$Y \leq X \leq X_0,$$

where $c, \delta$ are positive constants. When $\beta = 0$ in (7), we obtain

$$\Lambda = x_1 \vartheta_1 + x_2 \vartheta_2.$$

Put $\vartheta = -\vartheta_1 / \vartheta_2$. We assume that $(x_1, x_2) = 1$. Let the continued fraction expansion of $\vartheta$ be given by

$$[a_0, a_1, a_2, \ldots],$$

and let the $k$-th convergent of $\vartheta$ be $p_k / q_k$ for $k = 0, 1, 2, \ldots$. Without loss of generality, we assume that $|\vartheta_1| < |\vartheta_2|$ and $x_1 > 0$. We have the following results.

Lemma 8 (de Weger [20]). Let

$$A = \max_{0 \leq k \leq \gamma_0} a_{k+1},$$
where

\[ Y_0 = -1 + \frac{\log(\sqrt{5}X_0 + 1)}{\log\left(\frac{1+\sqrt{5}}{2}\right)}. \]

If (8) and (9) hold for \(x_1, x_2\) and \(\beta = 0\), then

\[ Y < \frac{1}{\delta} \log\left(\frac{c(A+2)X_0}{|\vartheta_2|}\right). \]

When \(\beta \neq 0\) in (7), put \(\vartheta = -\vartheta_1/\vartheta_2\) and \(\psi = \beta/\vartheta_2\). Then

\[ \frac{\Lambda}{\vartheta_2} = \psi - x_1 \vartheta + x_2. \]

Let \(p/q\) be a convergent of \(\vartheta\) with \(q > X_0\). For a real number \(x\), we write \(|x| = \min\{|x-n|, n \in \mathbb{Z}\}\) for the distance from \(x\) to the nearest integer. We may use the following Davenport Lemma.

**Lemma 9** (de Weger [20]). Suppose that

\[ \|q \psi\| > \frac{2X_0}{q}. \]

Then, the solutions of (8) and (9) satisfy

\[ Y < \frac{1}{\delta} \log\left(\frac{q^2 c}{|\vartheta_2|X_0}\right). \]

To apply Lemma 7 the following lemma will help us later to obtain inequality similar to (8).

**Lemma 10** (de Weger [20]). Let \(a, x \in \mathbb{R}\) and \(0 < a < 1\). If \(|x| < a\), then

\[ |\log(1 + x)| < \frac{-\log(1 - a)}{a}|x| \]

and

\[ |x| < \frac{a}{1 - e^{-a}}|e^x - 1|. \]

### 3. Proof of Theorem 1

It is easy to check that the only solutions are the ones listed in Theorem 1 if \(m \leq 3\). Thus, we shall suppose that \(m \geq 4\). By using Lemma 6 (for \(p = 3\)) together with Lemma 4, we derive that

\[ \frac{m}{4 \times 3} = \frac{m}{12} < v_3(m!!) = v_3(G_n) < v_3((n+1)(n+2)(n+4)(n+9) + 6 \leq 4v_3(n + \delta) + 6 \]

for some \(\delta \in \{1, 2, 4, 9\}\). Therefore, \(v_3(n + \delta) \geq \lambda\), where

\[ \lambda = \left\lfloor \frac{m - 72}{48} \right\rfloor. \]

Consequently, \(3^\lambda \mid n + \delta\). In particular, \(3^\lambda \leq n + \delta \leq n + 9\). Thus,

\[ \lambda \leq \frac{\log(n + 9)}{\log 3}. \quad (10) \]
Again, from Lemma 1,
\[(1.46)^{n-2} \leq G_n = m! < \left(\frac{m}{2}\right)^m.\]

Thus, \(n < 2.65m \log\left(\frac{m}{2}\right) + 2\). Substituting this in (10), we arrive at
\[
\lambda = \left\lfloor \frac{m - 72}{48} \right\rfloor \leq \frac{\log(2.65m \log\left(\frac{m}{2}\right) + 11)}{\log 3}.
\]

This inequality yields \(m \leq 503\) and then \(n < 2.65 \cdot 503 \cdot \log\left(\frac{503}{2}\right) + 2 = 7369.805 \cdots\). Now, we use a simple routine written in Mathematica that does not return any solution in the range \(4 \leq m \leq 503\) and \(1 \leq n \leq 7369\). The proof is complete. \(\square\)

4. Proof of Theorem 2
4.1. Absolute Bounds on Variables
In this section, we will use Baker’s method and the \(p\)-adic valuation to completely prove Theorem 2.

First, we give an upper bound for \(\ell\).

Lemma 11. If Diophantine Equation (1) has solutions, then \(\ell \leq 6\).

Proof. For all \(1 \leq a \leq 9\),
\[v_2\left(a \left(\frac{10^m - 1}{9}\right)\right) = v_2(a) \leq 3.\]

However, if \(\ell \geq 7\), then \(v_2(G_n G_{n+1} \cdots G_{n+(\ell-1)}) \geq 4\) by Corollary 2. \(\square\)

Next, we present an upper bound for \(n\) and \(m\).

Lemma 12. If \((n, \ell, m, a)\) is a positive integer solution of (1) with \(n \geq 15\), \(m \geq 2, 1 \leq a \leq 9\), and \(1 \leq \ell \leq 6\), then
\[
m \leq \ell n + \ell (\ell - 3)/2 \quad \text{and} \quad n < 1.86 \times 10^{16}.
\]

Proof. By (1) and (4), we have
\[
10^{n-1} < a \left(\frac{10^m - 1}{9}\right) = G_n G_{n+1} \cdots G_{n+(\ell-1)} < a^{\ell n + \ell (\ell - 3)/2} < 10^{\ell n + \ell (\ell - 3)/2}.
\]

Thus, we have
\[
m \leq \ell n + \frac{\ell (\ell - 3)}{2}.
\]

Now, by (3), we obtain
\[
G_n \cdots G_{n+(\ell-1)} = (c_\alpha a^n + e_n) \cdots (c_\alpha a^{n+(\ell-1)} + e_{n+(\ell-1)})
\]
\[
= c_\alpha^{\ell} a^{n+\cdots+n+(\ell-1)} + r_1(c_\alpha, \alpha, n, \ell)
\]
\[
= c_\alpha^{\ell} a^{n+\frac{\ell (\ell - 1)}{2}} + r_1(c_\alpha, \alpha, n, \ell),
\]

where \(r_1(c_\alpha, \alpha, n, \ell)\) involves the part of the expansion of the previous line that contains the product of powers of \(c_\alpha, \alpha\) and the errors \(e_i\) for \(i = n, \ldots, n + (\ell + 1)\). Moreover, \(r_1(c_\alpha, \alpha, n, \ell)\) is the sum of 63 terms with maximum absolute value \(c_\alpha^{\ell - 1} a^{n+\ell (\ell - 1)/2} n^{-\ell/2}\). Therefore, equality (11) enables us to express (1) in the form
\[
\frac{a}{9} 10^{m-1} c_\alpha^{\ell} a^{n+\frac{\ell (\ell - 1)}{2}} = \frac{a}{9} + r_1(c_\alpha, \alpha, n, \ell).
\]
Dividing both sides of the above equality by \( c_\alpha^{\ell-1} a^{(\ell-1)n + \ell(\ell-1)/2} \alpha^{-n/2} \) and taking the absolute value, we deduce that

\[
|H| \leq \left( \frac{a}{g} + |r_1(c_\alpha, n, \ell)| \right) \cdot c_\alpha^{-\ell} \alpha^{-(\ell n + \ell(\ell-1)/2)},
\]

where

\[
H = \frac{a}{9c_\alpha} a^{-(\ell n + \ell(\ell-1)/2)} 10^m - 1,
\]

and

\[
|r_1(c_\alpha, n, \ell)| \leq 63c_\alpha^{\ell-1} a^{(\ell-1)n + \ell(\ell-1)/2} \alpha^{-\frac{n}{2}}.
\]

Hence,

\[
|H| < (1 + 63c_\alpha^{\ell-1} a^{(\ell-1)n + \ell(\ell-1)/2} \alpha^{-\frac{n}{2}}) \cdot c_\alpha^{-\ell} \alpha^{-(\ell n + \ell(\ell-1)/2)} < c_\alpha^{-\ell} a^{-(\ell n + \ell(\ell-1)/2)} + 63c_\alpha^{\ell-1} a^{-\frac{n}{2}} \alpha^{-\frac{n}{2}} \leq 64c_\alpha^{\ell-1} a^{-\frac{n}{2}} < 105a^{-\frac{3n}{2}}. \tag{12}
\]

To find a lower bound for \( H \), we take \( z := 3 \),

\[
(\gamma_1, b_1) := \left( \frac{a}{9c_\alpha}, 1 \right), \quad (\gamma_2, b_2) := \left( a, -(\ell n + \ell(\ell-1)/2) \right), \quad \text{and} \quad (\gamma_3, b_3) := (10, m)
\]

in Lemma 7. For our choices, we have \( \gamma_1, \gamma_2, \gamma_3 \in \mathbb{Q}(a) \), with degree \( D := 3 \). One can see that \( H \neq 0 \). Otherwise, we obtain

\[
\frac{a}{9} 10^m = c_\alpha^{\ell} a^{-(\ell n + \ell(\ell-1)/2)}.
\]

Take the Galois automorphism \( \sigma := (\alpha \beta) \) and absolute values on both sides of the resulting equality, and we obtain

\[
1 < \frac{a}{9} 10^m = |c_\beta|^{\ell} |\beta|^{(\ell n + \ell(\ell-1)/2)} < 1.
\]

It is a contradiction. Thus, \( H \neq 0 \).

Now, we give estimates to \( A_i \) for \( i = 1, 2, 3 \). By the properties of the absolute logarithmic height, we have

\[
h(\gamma_1) \leq h(d) + h(9) + \ell h(c_\alpha) \leq 2 \log 9 + \ell h(c_\alpha).
\]

The minimal polynomial of \( c_\alpha \) is \( 31X^3 - 31X^2 + 9X - 1 \). Therefore, \( h(c_\alpha) = \frac{1}{3} \log 31 \) and thus

\[
h(\gamma_1) \leq 2 \log 9 + 2 \log 31.
\]

On the other hand, \( h(\gamma_2) = \frac{1}{3} \log \alpha \) and \( h(\gamma_3) = \log 10 \). Thus, we take

\[
A_1 := 34, \quad A_2 := 0.4, \quad \text{and} \quad A_3 := 7.
\]

By (11) and the fact \( \ell \leq 6 \), we take \( B := 6n + 15 \). Applying Lemma 7, we obtain a lower bound for \( |H| \), which, by comparing it to (12), leads to

\[
\frac{3n}{2} \log \alpha - \log 105 < C(s, D)(1 + \log B)A_1A_2A_3 = 2.57 \times 10^{14} (1 + \log(6n + 5)).
\]
Therefore, we obtain
\[
n < 4.6 \times 10^{14} (1 + \log(6n + 5)).
\]
Hence, we obtain \( n < 1.86 \times 10^{16} \).

4.2. Reducing the Bounds

Next, we try to lower the bound of \( n \); we will use Lemma 9. Let
\[
\Lambda_1 := m \log 10 - \left( \ell n + \frac{\ell(\ell - 1)}{2} \right) \log \alpha + \log \left( \frac{a}{9c^\ell} \right).
\]

Therefore, (12) can be written as
\[
|e^{\Lambda_1} - 1| < 105\alpha^{-\frac{3n}{2}}.
\]
Furthermore,
\[
|H| < 0.02, \quad \text{if } n \geq 15.
\]
Therefore, by applying Lemma 10, we deduce that
\[
|\Lambda_1| < -\frac{\log(1 - 0.02)}{0.02} |H| < 10^7 \exp(-0.57n).
\]

Put
\[
\theta_1 := -\log \alpha, \quad \theta_2 := 10, \quad \psi := \log \left( \frac{a}{9c^\ell} \right), \quad c := 107, \quad \delta := 0.57.
\]

Furthermore, as
\[
\max \left\{ m, \ell n + \frac{\ell(\ell - 1)}{2} \right\} < 1.12 \times 10^{17},
\]
we take
\[
X_0 = 1.12 \times 10^{17}.
\]

One can use Mathematica to see that
\[
q_{31} = 4488925465739399775
\]
satisfies the condition of Lemma 9 for all \( 1 \leq a \leq 9 \) and \( 1 \leq \ell \leq 6 \). Therefore, Lemma 9 implies that if the Diophantine Equation (1) has solutions, then
\[
n \leq \frac{1}{0.57} \times \log \left( \frac{4488925465739399775^2 \times 107}{\log 10 \times 1.12 \times 10^{17}} \right) < 89.
\]

Now, we reduce again this new bound of \( n \). In this application of Lemma 9, we find
\[
\max \left\{ m, \ell n + \frac{\ell(\ell - 1)}{2} \right\} < 543,
\]
and then we take \( X_0 = 543 \) and see that \( q_8 = 174565 \) satisfies the conditions of Lemma 9. Thus, we obtain
\[
n \leq \frac{1}{0.57} \times \log \left( \frac{174565^2 \times 107}{\log 10 \times 543} \right) < 39.
\]

Hence, it remains to check (1) for \( 1 \leq n \leq 38, 1 \leq \ell \leq 6, 2 \leq m \leq 237, 1 \leq a \leq 9 \). By a fast computation with Mathematica in these ranges, we conclude that the quadruple
\((n, \ell, m, a) = (13, 1, 2, 8)\) is the only solution of the Diophantine equation. This completes the proof of Theorem 2.

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