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5.7 Lengyel's Constant

5.7.1 Stirling Partition Numbers

Let S be a set with n elements. The set of all subsets of S has 2^n elements. By a **partition** of S we mean a disjoint set of nonempty subsets (called **blocks**) whose union is S. The set of partitions of S that possess exactly k blocks has $S_{n,k}$ elements, where $S_{n,k}$ is a

Stirling number of the second kind. The set of *all* partitions of S has B_n elements, where B_n is a **Bell number**:

$$B_n = \sum_{k=1}^{n} S_{n,k} = \frac{1}{e}\sum_{j=0}^{\infty}\frac{j^n}{j!} = \left.\frac{d^n}{dx^n}\exp(e^x - 1)\right|_{x=0}.$$

For example, $S_{4,1} = 1$, $S_{4,2} = 7$, $S_{4,3} = 6$, $S_{4,4} = 1$, and $B_4 = 15$. More generally, $S_{n,1} = 1$, $S_{n,2} = 2^{n-1} - 1$, and $S_{n,3} = \frac{1}{2}(3^{n-1} + 1) - 2^{n-1}$. The following recurrences are helpful [1–4]:

$$S_{n,0} = \begin{cases} 1 & \text{if } n = 0, \\ 0 & \text{if } n \geq 1, \end{cases} \quad S_{n,k} = kS_{n-1,k} + S_{n-1,k-1} \quad \text{if } n \geq k \geq 1,$$

$$B_0 = 1, \quad B_n = \sum_{k=0}^{n-1}\binom{n-1}{k}B_k.$$

and corresponding asymptotics are discussed in [5–9].

5.7.2 Chains in the Subset Lattice of S

If U and V are subsets of S, write $U \subset V$ if U is a proper subset of V. This endows the set of all subsets of S with a **partial ordering**; in fact, it is a **lattice** with maximum element S and minimum element $\emptyset$. The number of **chains** $\emptyset = U_0 \subset U_1 \subset \cdots \subset U_{k-1} \subset U_k = S$ of length k is $k!S_{n,k}$. Hence the number of all chains from $\emptyset$ to S is [1,6,10]

$$\sum_{k=0}^{n} k!S_{n,k} = \sum_{j=0}^{\infty}\frac{j^n}{2^{j+1}} = \frac{1}{2}\operatorname{Li}_{-n}\left(\frac{1}{2}\right) = \left.\frac{d^n}{dx^n}\frac{1}{2-e^x}\right|_{x=0} \sim \frac{n!}{2}\left(\frac{1}{\ln(2)}\right)^{n+1},$$

where $\operatorname{Li}_m(x)$ is the polylogarithm function. Wilf [10] marveled at how accurate this asymptotic approximation is.

If we further insist that the chains are **maximal**, equivalently, that additional proper insertions are impossible, then the number of such chains is $n!$ A general technique due to Doubilet, Rota & Stanley [11], involving what are called *incidence algebras*, can be used to obtain the two aforementioned results, as well as to enumerate chains within more complicated posets [12].

As an aside, we give a deeper application of incidence algebras: to enumerating chains of linear subspaces within finite vector spaces [6]. Define the q-**binomial coefficient** and q-**factorial** by

$$\binom{n}{k}_q = \frac{\prod_{j=1}^{n}(q^j - 1)}{\prod_{j=1}^{k}(q^j - 1)\cdot\prod_{j=1}^{n-k}(q^j - 1)},$$

$$[n!]_q = (1+q)(1+q+q^2)\cdots(1+q+\cdots+q^{n-1}),$$

where $q > 1$. Note the special case in the limit as $q \to 1^+$. Consider the n-dimensional vector space $\mathbb{F}_q^n$ over the finite field $\mathbb{F}_q$, where q is a prime power [12–16]. The number of k-dimensional linear subspaces of $\mathbb{F}_q^n$ is $\binom{n}{k}_q$ and the total number of linear subspaces of $\mathbb{F}_q^n$ is asymptotically $c_e q^{n^2/4}$ if n is even and $c_o q^{n^2/4}$ if n is odd, where [17, 18]

$$c_e = \frac{\sum_{k=-\infty}^{\infty} q^{-k^2}}{\prod_{j=1}^{\infty}(1-q^{-j})}, \quad c_o = \frac{\sum_{k=-\infty}^{\infty} q^{-\left(k+\frac{1}{2}\right)^2}}{\prod_{j=1}^{\infty}(1-q^{-j})}.$$

We give a recurrence for the number χ_n of chains of proper subspaces (again, ordered by inclusion):

$$\chi_1 = 1, \quad \chi_n = 1 + \sum_{k=1}^{n-1} \binom{n}{k}_q \chi_k \quad \text{for } n \geq 2.$$

For the asymptotics, it follows that [6, 17]

$$\chi_n \sim \frac{1}{\zeta_q'(r)r}\left(\frac{1}{r}\right)^n \prod_{j=1}^{n}(q^j - 1) = \frac{A}{r^n}(q-1)(q^2-1)(q^3-1)\cdots(q^n-1),$$

where $\zeta_q(x)$ is the zeta function for the poset of subspaces:

$$\zeta_q(x) = \sum_{k=1}^{\infty} \frac{x^k}{(q-1)(q^2-1)(q^3-1)\cdots(q^k-1)}$$

and $r > 0$ is the unique solution of the equation $\zeta_q(r) = 1$. In particular, when $q = 2$, we have $c_e = 7.3719688014\ldots$, $c_o = 7.3719494907\ldots$, and

$$\chi_n \sim \frac{A}{r^n} \cdot Q \cdot 2^{\frac{n(n+1)}{2}},$$

where $r = 0.7759021363\ldots$, $A = 0.8008134543\ldots$, and

$$Q = \prod_{k=1}^{\infty}\left(1 - \frac{1}{2^k}\right) = 0.2887880950\ldots$$

is one of the digital search tree constants [5.14]. If we further insist that the chains are maximal, then the number of such chains is $[n!]_q$.

5.7.3 Chains in the Partition Lattice of S

We have discussed chains in the poset of subsets of the set S. There is, however, another poset associated naturally with S that is less familiar and more difficult to study: the **poset of partitions** of S. Here is the partial ordering: Assuming P and Q are two partitions of S, then $P < Q$ if $P \neq Q$ and if $p \in P$ implies that p is a subset of q for some $q \in Q$. In other words, P is a *refinement* of Q in the sense that each of its blocks fits within a block of Q. For arbitrary n, the poset is, in fact, a lattice with minimum element $m = \{\{1\}, \{2\}, \ldots, \{n\}\}$ and maximum element $M = \{\{1, 2, \ldots, n\}\}$.

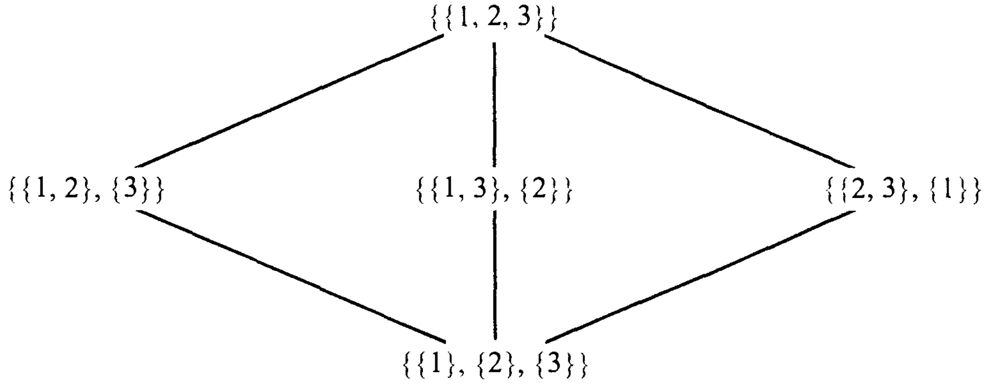

Figure 5.10. The number of chains $m < P_1 < M$ in the partition lattice of the set $\{1, 2, 3\}$ is three.

What is the number of chains $m = P_0 < P_1 < P_2 < \cdots < P_{k-1} < P_k = M$ of length k in the partition lattice of S? In the case $n = 3$, there is only one chain for $k = 1$, specifically, $m < M$. For $k = 2$, there are three such chains as pictured in Figure 5.10.

Let Z_n denote the number of all chains from m to M of any length; clearly $Z_1 = Z_2 = 1$ and, by the foregoing, $Z_3 = 4$. We have the recurrence

$$Z_n = \sum_{k=1}^{n-1} S_{n,k} Z_k$$

and exponential generating function

$$Z(x) = \sum_{n=1}^{\infty} \frac{Z_n}{n!} x^n, \quad 2Z(x) = x + Z(e^x - 1),$$

but techniques of Doubilet, Rota & Stanley and Bender do not apply here to give asymptotic estimates of Z_n. The partition lattice is the first natural lattice without the structure of a *binomial lattice*, which implies that well-known generating function techniques are no longer helpful.

Lengyel [19] formulated a different approach to prove that the quotient

$$r_n = \frac{Z_n}{(n!)^2 (2\ln(2))^{-n} n^{-1-\ln(2)/3}}$$

must be bounded between two positive constants as $n \to \infty$. He presented numerical evidence suggesting that r_n tends to a unique value. Babai & Lengyel [20] then proved a fairly general convergence criterion that enabled them to conclude that $\Lambda = \lim_{n\to\infty} r_n$ exists and $\Lambda = 1.09\ldots$. The analysis in [19] involves intricate estimates of the Stirling numbers; in [20], the focus is on nearly convex linear recurrences with finite retardation and active predecessors.

In an ambitious undertaking, Flajolet & Salvy [21] computed $\Lambda = 1.0986858055\ldots$. Their approach is based on (complex fractional) analytic iterates of $\exp(x) - 1$ and much more, but unfortunately their paper is presently incomplete. See [5.8] for related discussion of the Takeuchi-Prellberg constant.

By way of contrast, the number of *maximal* chains is given exactly by $n!(n-1)!/2^{n-1}$ and Lengyel [19] observed that Z_n exceeds this by an exponentially large factor.

5.7.4 Random Chains

Van Cutsem & Ycart [22] examined random chains in both the subset and partition lattices. It is remarkable that a common framework exists for studying these and that, in a certain sense, the limiting distributions of both types of chains are *identical.* We mention only one consequence: If $\kappa_n = k/n$ is the normalized length of the random chain, then

$$\lim_{n\to\infty} \mathrm{E}(\kappa_n) = \frac{1}{2\ln(2)} = 0.7213475204\ldots$$

and a corresponding Central Limit Theorem also holds.

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5.8 Takeuchi–Prellberg Constant

In 1978, Takeuchi defined a triply recursive function [1,2]

$$t(x,y,z) = \begin{cases} y & \text{if } x \le y, \\ t(t(x-1,y,z), t(y-1,z,x), t(z-1,x,y)) & \text{otherwise} \end{cases}$$

that is useful for benchmark testing of programming languages. The value of $t(x,y,z)$ is of no practical significance; in fact, McCarthy [1,2] observed that the function can be described more simply as

$$t(x,y,z) = \begin{cases} y & \text{if } x \le y, \\ \begin{cases} z & \text{if } y \le z, \\ x & \text{otherwise,} \end{cases} & \text{otherwise.} \end{cases}$$

The interesting quantity is not $t(x,y,z)$, but rather $T(x,y,z)$, defined to be the number of times the *otherwise* clause is invoked in the recursion. We assume that the program is memoryless in the sense that previously computed results are not available at any time in the future. Knuth [1,3] studied the **Takeuchi numbers** $T_n = T(n, 0, n+1)$:

$$T_0 = 0, \;\; T_1 = 1, \;\; T_2 = 4, \;\; T_3 = 14, \;\; T_4 = 53, \;\; T_5 = 223, \ldots$$

and deduced that

$$e^{n\ln(n) - n\ln(\ln(n)) - n} < T_n < e^{n\ln(n) - n + \ln(n)}$$

for all sufficiently large n. He asked for more precise asymptotic information about the growth of T_n.

Starting with Knuth's recursive formula for the Takeuchi numbers

$$T_{n+1} = \sum_{k=0}^{n} \left[\binom{n+k}{n} - \binom{n+k}{n+1}\right] T_{n-k} + \sum_{k=1}^{n-1} \binom{2k}{k} \frac{1}{k+1}$$

and the somewhat related Bell numbers [5.7]

$$B_{n+1} = \sum_{k=0}^{n} \binom{n}{k} B_{n-k}, \;\; B_0 = 1, \;\; B_1 = 1, \;\; B_2 = 2, \;\; B_3 = 5, \;\; B_4 = 15, \;\; B_5 = 52, \ldots.$$

Prellberg [4] observed that the following limit exists:

$$c = \lim_{n\to\infty} \frac{T_n}{B_n \exp\left(\frac{1}{2} W_n^2\right)} = 2.2394331040\ldots,$$

where $W_n \exp(W_n) = n$ are special values of the Lambert W function [6.11].

Since both the Bell numbers and the W function are well understood, this provides an answer to Knuth's question. The underlying theory is still under development, but

Prellberg's numerical evidence is persuasive. Recent theoretical work [5] relates the constant c to an associated functional equation,

$$T(z) = \sum_{n=0}^{\infty} T_n z^n, \quad T(z) = \frac{T(z - z^2)}{z} - \frac{1}{(1 - z)(1 - z + z^2)},$$

in a manner parallel to how Lengyel's constant [5.7] is obtained.

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5.9 Pólya's Random Walk Constants

Let L denote the d-dimensional cubic lattice whose vertices are precisely all integer points in d-dimensional space. A **walk** ω on L, beginning at the origin, is an infinite sequence of vertices $\omega_0, \omega_1, \omega_2, \omega_3, \ldots$ with $\omega_0 = 0$ and $|\omega_{j+1} - \omega_j| = 1$ for all j. Assume that the walk is random and symmetric in the sense that, at each time step, all $2d$ directions of possible travel have equal probability. What is the likelihood that $\omega_n = 0$ for some $n > 0$? That is, what is the **return probability** p_d?

Pólya [1–4] proved the remarkable fact that $p_1 = p_2 = 1$ but $p_d < 1$ for $d > 2$. McCrea & Whipple [5], Watson [6], Domb [7] and Glasser & Zucker [8] each contributed facets of the following evaluations of $p_3 = 1 - 1/m_3 = 0.3405373295\ldots$, where the expected number m_3 of returns to the origin, plus one, is

$$\begin{aligned} m_3 &= \frac{3}{(2\pi)^3} \int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} \frac{1}{3 - \cos(\theta) - \cos(\varphi) - \cos(\psi)}\, d\theta\, d\varphi\, d\psi \\ &= \frac{12}{\pi^2}\left(18 + 12\sqrt{2} - 10\sqrt{3} - 7\sqrt{6}\right) K\left[(2 - \sqrt{3})(\sqrt{3} - \sqrt{2})\right]^2 \\ &= 3\left(18 + 12\sqrt{2} - 10\sqrt{3} - 7\sqrt{6}\right)\left[1 + 2\sum_{k=1}^{\infty} \exp(-\sqrt{6}\pi k^2)\right]^4 \\ &= \frac{\sqrt{6}}{32\pi^3}\Gamma\left(\frac{1}{24}\right)\Gamma\left(\frac{5}{24}\right)\Gamma\left(\frac{7}{24}\right)\Gamma\left(\frac{11}{24}\right) = 1.5163860591\ldots. \end{aligned}$$

Hence the **escape probability** for a random walk on the three-dimensional cubic lattice is $1 - p_3 = 0.6594626704\ldots$. In these expressions, K denotes the complete elliptic integral of the first kind [1.4.6] and Γ denotes the gamma function [1.5.4]. Return and escape probabilities can also be computed for the body-centered or face-centered cubic