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Mathematical Constants

STEVEN R. FINCH

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Well-Known Constants

1.1 Pythagoras' Constant, $\sqrt{2}$

The diagonal of a unit square has length $\sqrt{2} = 1.4142135623\ldots$. A theory, proposed by the Pythagorean school of philosophy, maintained that all geometric magnitudes could be expressed by rational numbers. The sides of a square were expected to be commensurable with its diagonals, in the sense that certain integer multiples of one would be equivalent to integer multiples of the other. This theory was shattered by the discovery that $\sqrt{2}$ is irrational [1–4].

Here are two proofs of the irrationality of $\sqrt{2}$, the first based on divisibility properties of the integers and the second using well ordering.

- If $\sqrt{2}$ were rational, then the equation $p^2 = 2q^2$ would be solvable in integers p and q, which are assumed to be in lowest terms. Since p^2 is even, p itself must be even and so has the form $p = 2r$. This leads to $2q^2 = 4r^2$ and thus q must also be even. But this contradicts the assumption that p and q were in lowest terms.
- If $\sqrt{2}$ were rational, then there would be a least positive integer s such that $s\sqrt{2}$ is an integer. Since $1 < 2$, it follows that $1 < \sqrt{2}$ and thus $t = s \cdot (\sqrt{2} - 1)$ is a positive integer. Also $t\sqrt{2} = s \cdot (\sqrt{2} - 1)\sqrt{2} = 2s - s\sqrt{2}$ is an integer and clearly $t < s$. But this contradicts the assumption that s was the smallest such integer.

Newton's method for solving equations gives rise to the following first-order recurrence, which is very fast and often implemented:

$$x_0 = 1, \qquad x_k = \frac{x_{k-1}}{2} + \frac{1}{x_{k-1}} \quad \text{for } k \geq 1, \qquad \lim_{k\to\infty} x_k = \sqrt{2}.$$

Another first-order recurrence [5] yields the reciprocal of $\sqrt{2}$:

$$y_0 = \frac{1}{2}, \qquad y_k = y_{k-1}\left(\frac{3}{2} - y_{k-1}^2\right) \quad \text{for } k \geq 1, \qquad \lim_{k\to\infty} y_k = \frac{1}{\sqrt{2}}.$$

The binomial series, also due to Newton, provides two interesting summations [6]:

$$1+\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{2^{2n}(2n-1)}\binom{2n}{n}=1+\frac{1}{2}-\frac{1}{2\cdot 4}+\frac{1\cdot 3}{2\cdot 4\cdot 6}-+\cdots=\sqrt{2},$$

$$1+\sum_{n=1}^{\infty}\frac{(-1)^{n}}{2^{2n}}\binom{2n}{n}=1-\frac{1}{2}+\frac{1\cdot 3}{2\cdot 4}-\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}+-\cdots=\frac{1}{\sqrt{2}}.$$

The latter is extended in [1.5.4]. We mention two beautiful infinite products [5, 7, 8]

$$\prod_{n=1}^{\infty}\left(1+\frac{(-1)^{n+1}}{2n-1}\right)=\left(1+\frac{1}{1}\right)\left(1-\frac{1}{3}\right)\left(1+\frac{1}{5}\right)\left(1-\frac{1}{7}\right)\cdots=\sqrt{2},$$

$$\prod_{n=1}^{\infty}\left(1-\frac{1}{4(2n-1)^2}\right)=\frac{1\cdot 3}{2\cdot 2}\cdot\frac{5\cdot 7}{6\cdot 6}\cdot\frac{9\cdot 11}{10\cdot 10}\cdot\frac{13\cdot 15}{14\cdot 14}\cdots=\frac{1}{\sqrt{2}}$$

and the regular continued fraction [9]

$$2+\cfrac{1}{2+\cfrac{1}{2+\cfrac{1}{2+\cdots}}}=2+\frac{1|}{|2}+\frac{1|}{|2}+\frac{1|}{|2}+\cdots=1+\sqrt{2}=(-1+\sqrt{2})^{-1},$$

which is related to **Pell's sequence**

$$a_0=0,\qquad a_1=1,\qquad a_n=2a_{n-1}+a_{n-2}\quad \text{for } n\geq 2$$

via the limiting formula

$$\lim_{n\to\infty}\frac{a_{n+1}}{a_n}=1+\sqrt{2}.$$

This is completely analogous to the famous connection between the Golden mean φ and Fibonacci's sequence [1.2]. See also Figure 1.1.

Viète's remarkable product for Archimedes' constant π [1.4.2] involves only the number 2 and repeated square-root extractions. Another expression connecting π and radicals appears in [1.4.5].

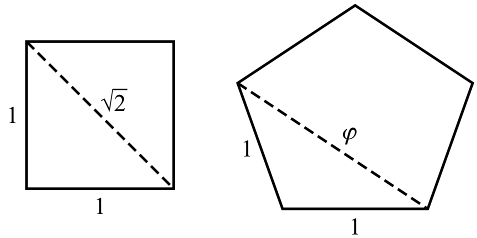

Figure 1.1. The diagonal of a regular unit pentagon, connecting any two nonadjacent corners, has length given by the Golden mean φ (rather than by Pythagoras' constant).

We return finally to irrationality issues: There obviously exist rationals x and y such that x^y is irrational (just take $x = 2$ and $y = 1/2$). Do there exist *irrationals* x and y such that x^y is *rational*? The answer to this is very striking. Let

$$z = \sqrt{2}^{\sqrt{2}}.$$

If z is rational, then take $x = y = \sqrt{2}$. If z is irrational, then take $x = z$ and $y = \sqrt{2}$, and clearly $x^y = 2$. Thus we have answered the question ("yes") without addressing the actual arithmetical nature of z. In fact, z is transcendental by the Gel'fond–Schneider theorem [10], proved in 1934, and hence is irrational. There are many unsolved problems in this area of mathematics; for example, we do not know whether

$$\sqrt{2}^{z} = \sqrt{2}^{\sqrt{2}^{\sqrt{2}}}$$

is irrational (let alone transcendental).

1.1.1 Generalized Continued Fractions

It is well known that any quadratic irrational possesses a periodic regular continued fraction expansion and vice versa. Comparatively few people have examined the generalized continued fraction [11–17]

$$w(p,q) = q + \cfrac{p + \cfrac{1}{q + \cfrac{p + \cdots}{q + \cdots}}}{q + \cfrac{p + \cfrac{1 + \cdots}{q + \cdots}}{q + \cfrac{p + \cdots}{q + \cdots}}},$$

which exhibits a fractal-like construction. Each *new* term in a particular generation (i.e., in a partial convergent) is replaced according to the rules

$$p \to p + \frac{1}{q}, \qquad q \to q + \frac{p}{q}$$

in the next generation. Clearly

$$w = q + \frac{p + \frac{1}{w}}{w}; \qquad \text{that is,} \quad w^3 - qw^2 - pw - 1 = 0.$$

In the special case $p = q = 3$, the higher-order continued fraction converges to $(-1 + \sqrt[3]{2})^{-1}$. It is conjectured that regular continued fractions for cubic irrationals behave like those for almost all real numbers [18–21], and no patterns are evident. The ordinary replacement rule

$$r \to r + \frac{1}{r}$$

is sufficient for the study of quadratic irrationals, but requires extension for broader classes of algebraic numbers.

Two alternative representations of $\sqrt[3]{2}$ are as follows [22]:

$$\sqrt[3]{2} = 1 + \cfrac{1}{3 + \frac{3}{a} + \frac{1}{b}}, \qquad \text{where} \quad a = 3 + \frac{3}{a} + \frac{1}{b}, \qquad b = 12 + \frac{10}{a} + \frac{3}{b}$$

and [23]

$$\sqrt[3]{2} = 1 + \frac{1|}{|3} + \frac{2|}{|2} + \frac{4|}{|9} + \frac{5|}{|2} + \frac{7|}{|15} + \frac{8|}{|2} + \frac{10|}{|21} + \frac{11|}{|2} + \cdots.$$

Other usages of the phrase "generalized continued fractions" include those in [24], with application to simultaneous Diophantine approximation, and in [25], with a geometric interpretation involving the boundaries of convex hulls.

1.1.2 Radical Denestings

We mention two striking radical denestings due to Ramanujan:

$$\sqrt[3]{\sqrt[3]{2} - 1} = \sqrt[3]{\frac{1}{9}} - \sqrt[3]{\frac{2}{9}} + \sqrt[3]{\frac{4}{9}}, \qquad \sqrt{\sqrt[3]{5} - \sqrt[3]{4}} = \tfrac{1}{3}\left(\sqrt[3]{2} + \sqrt[3]{20} - \sqrt[3]{25}\right).$$

Such simplifications are an important part of computer algebra systems [26].

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1.2 The Golden Mean, φ

Consider a line segment:

What is the most “pleasing” division of this line segment into two parts? Some people might say at the halfway point:

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Others might say at the one-quarter or three-quarters point. The “correct answer” is, however, none of these, and is supposedly found in Western art from the ancient Greeks onward (aestheticians speak of it as the principle of “dynamic symmetry”):

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If the right-hand portion is of length $v = 1$, then the left-hand portion is of length $u = 1.618\ldots$. A line segment partitioned as such is said to be divided in Golden or Divine section. What is the justification for endowing this particular division with such elevated status? The length u, as drawn, is to the whole length $u + v$, as the length v is to u:

$$\frac{u}{u+v} = \frac{v}{u}.$$

Letting $\varphi = u/v$, solve for φ via the observation that

$$1 + \frac{1}{\varphi} = 1 + \frac{v}{u} = \frac{u+v}{u} = \frac{u}{v} = \varphi.$$

The positive root of the resulting quadratic equation $\varphi^2 - \varphi - 1 = 0$ is

$$\varphi = \frac{1+\sqrt{5}}{2} = 1.6180339887\ldots,$$

which is called the **Golden mean** or **Divine proportion** [1,2].

The constant φ is intricately related to **Fibonacci's sequence**

$$f_0 = 0, \qquad f_1 = 1, \qquad f_n = f_{n-1} + f_{n-2} \quad \text{for } n \geq 2.$$

This sequence models (in a naive way) the growth of a rabbit population. Rabbits are assumed to start having bunnies once a month after they are two months old; they always give birth to twins (one male bunny and one female bunny), they never die, and they never stop propagating. The number of rabbit pairs after n months is f_n.

What can φ possibly have in common with $\{f_n\}$? This is one of the most remarkable ideas in all of mathematics. The partial convergents leading up to the regular continued fraction representation of φ,

$$\varphi = 1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cdots}}} = 1 + \frac{1|}{|1} + \frac{1|}{|1} + \frac{1|}{|1} + \cdots,$$

are all ratios of successive Fibonacci numbers; hence

$$\lim_{n\to\infty} \frac{f_{n+1}}{f_n} = \varphi.$$

This result is also true for arbitrary sequences satisfying the same recursion $f_n = f_{n-1} + f_{n-2}$, assuming that the initial terms f_0 and f_1 are distinct [3,4].

The rich geometric connection between the Golden mean and Fibonacci's sequence is seen in Figure 1.2. Starting with a single Golden rectangle (of length φ and width 1), there is a natural sequence of nested Golden rectangles obtained by removing the leftmost square from the first rectangle, the topmost square from the second rectangle, etc. The length and width of the n^{th} Golden rectangle can be written as linear expressions $a + b\varphi$, where the coefficients a and b are always Fibonacci numbers. These Golden rectangles can be inscribed in a logarithmic spiral as pictured. Assume that the lower left corner of the first rectangle is the origin of an xy-coordinate system.

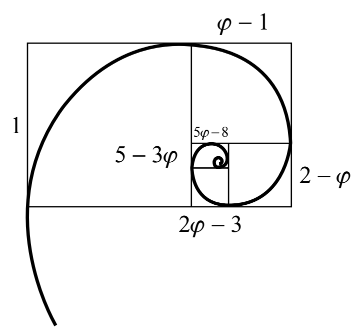

Figure 1.2. The Golden spiral circumscribes the sequence of Golden rectangles.

The accumulation point for the spiral can be proved to be $(\frac{1}{5}(1+3\varphi), \frac{1}{5}(3-\varphi))$. Such logarithmic spirals are "equiangular" in the sense that every line through (x_∞, y_∞) cuts across the spiral at a constant angle ξ. In this way, logarithmic spirals generalize ordinary circles (for which $\xi = 90°$). The logarithmic spiral pictured gives rise to the constant angle $\xi = \operatorname{arccot}(\frac{2}{\pi}\ln(\varphi)) = 72.968\ldots°$. Logarithmic spirals are evidently found throughout nature; for example, the shell of a chambered nautilus, the tusks of an elephant, and patterns in sunflowers and pine cones [4–6].

Another geometric application of the Golden mean arises when inscribing a regular pentagon within a given circle by ruler and compass. This is related to the fact that

$$2\cos\left(\frac{\pi}{5}\right) = \varphi, \qquad 2\sin\left(\frac{\pi}{5}\right) = \sqrt{3-\varphi}.$$

The Golden mean, just as it has a simple regular continued fraction expansion, also has a simple radical expansion [7]

$$\varphi = \sqrt{1+\sqrt{1+\sqrt{1+\sqrt{1+\sqrt{1+\cdots}}}}}.$$

The manner in which this expansion converges to φ is discussed in [1.2.1]. Like Pythagoras' constant [1.1], the Golden mean is irrational and simple proofs are given in [8, 9].

Here is a series [10] involving φ:

$$\frac{2\sqrt{5}}{5}\ln(\varphi) = \left(1-\frac{1}{2}-\frac{1}{3}+\frac{1}{4}\right)+\left(\frac{1}{6}-\frac{1}{7}-\frac{1}{8}+\frac{1}{9}\right) + \left(\frac{1}{11}-\frac{1}{12}-\frac{1}{13}+\frac{1}{14}\right)+\cdots,$$

which reminds us of certain series connected with Archimedes' constant [1.4.1]. A direct expression for φ as a sum can be obtained from the Taylor series for the square root function, expanded about 4. The Fibonacci numbers appear in yet another representation [11] of φ:

$$4-\varphi = \sum_{n=0}^{\infty}\frac{1}{f_{2^n}} = \frac{1}{f_1}+\frac{1}{f_2}+\frac{1}{f_4}+\frac{1}{f_8}+\cdots.$$

Among many other possible formulas involving φ, we mention the four Rogers–Ramanujan continued fractions

$$\frac{1}{\alpha-\varphi}\exp\left(-\frac{2\pi}{5}\right) = 1+\frac{e^{-2\pi}|}{|1}+\frac{e^{-4\pi}|}{|1}+\frac{e^{-6\pi}|}{|1}+\frac{e^{-8\pi}|}{|1}+\cdots,$$

$$\frac{1}{\beta-\varphi}\exp\left(-\frac{2\pi}{\sqrt{5}}\right) = 1+\frac{e^{-2\pi\sqrt{5}}|}{|1}+\frac{e^{-4\pi\sqrt{5}}|}{|1}+\frac{e^{-6\pi\sqrt{5}}|}{|1}+\frac{e^{-8\pi\sqrt{5}}|}{|1}+\cdots,$$

$$\frac{1}{\kappa-(\varphi-1)}\exp\left(-\frac{\pi}{5}\right) = 1-\frac{e^{-\pi}|}{|1}+\frac{e^{-2\pi}|}{|1}-\frac{e^{-3\pi}|}{|1}+\frac{e^{-4\pi}|}{|1}-+\cdots,$$

$$\frac{1}{\lambda-(\varphi-1)}\exp\left(-\frac{\pi}{\sqrt{5}}\right) = 1-\frac{e^{-\pi\sqrt{5}}|}{|1}+\frac{e^{-2\pi\sqrt{5}}|}{|1}-\frac{e^{-3\pi\sqrt{5}}|}{|1}+\frac{e^{-4\pi\sqrt{5}}|}{|1}-+\cdots,$$

where

$$\alpha = \left(\varphi\sqrt{5}\right)^{\frac{1}{2}}, \qquad \alpha' = \frac{1}{\sqrt{5}}\left((\varphi-1)\sqrt{5}\right)^{\frac{5}{2}}, \qquad \beta = \frac{\sqrt{5}}{1+\sqrt[5]{\alpha'-1}},$$
$$\kappa = \left((\varphi-1)\sqrt{5}\right)^{\frac{1}{2}}, \qquad \kappa' = \frac{1}{\sqrt{5}}\left(\varphi\sqrt{5}\right)^{\frac{5}{2}}, \qquad \lambda = \frac{\sqrt{5}}{1+\sqrt[5]{\kappa'-1}}.$$

The fourth evaluation is due to Ramanathan [9, 12–16].

1.2.1 Analysis of a Radical Expansion

The radical expansion [1.2] for φ can be rewritten as a sequence $\{\varphi_n\}$:

$$\varphi_1 = 1, \qquad \varphi_n = \sqrt{1+\varphi_{n-1}} \quad \text{for } n \geq 2.$$

Paris [17] proved that the rate in which φ_n approaches the limit φ is given by

$$\varphi - \varphi_n \sim \frac{2C}{(2\varphi)^n} \quad \text{as } n \to \infty,$$

where $C = 1.0986419643\ldots$ is a new constant. Here is an exact characterization of C. Let $F(x)$ be the analytic solution of the functional equation

$$F(x) = 2\varphi F(\varphi - \sqrt{\varphi^2 - x}), \qquad |x| < \varphi^2,$$

subject to the initial conditions $F(0) = 0$ and $F'(0) = 1$. Then $C = \varphi F(1/\varphi)$. A power-series technique can be used to evaluate C numerically from these formulas. It is simpler, however, to use the following product:

$$C = \prod_{n=2}^{\infty} \frac{2\varphi}{\varphi + \varphi_n},$$

which is stable and converges quickly [18].

Another interesting constant is defined via the radical expression [7, 19]

$$\sqrt{1+\sqrt{2+\sqrt{3+\sqrt{4+\sqrt{5+\cdots}}}}} = 1.7579327566\ldots,$$

but no expression of this in terms of other constants is known.

1.2.2 Cubic Variations of the Golden Mean

Perrin's sequence is defined by

$$g_0 = 3, \qquad g_1 = 0, \qquad g_2 = 2, \qquad g_n = g_{n-2} + g_{n-3} \quad \text{for } n \geq 3$$

and has the property that $n > 1$ divides g_n if n is prime [20, 21]. The limit of ratios of successive Perrin numbers

$$\psi = \lim_{n\to\infty} \frac{g_{n+1}}{g_n}$$

satisfies $\psi^3 - \psi - 1 = 0$ and is given by

$$\psi = \left(\tfrac{1}{2} + \tfrac{\sqrt{69}}{18}\right)^{\frac{1}{3}} + \tfrac{1}{3}\left(\tfrac{1}{2} + \tfrac{\sqrt{69}}{18}\right)^{-\frac{1}{3}} = \tfrac{2\sqrt{3}}{3}\cos\left(\tfrac{1}{3}\arccos\left(\tfrac{3\sqrt{3}}{2}\right)\right)$$
$$= 1.3247179572\ldots.$$

This also has the radical expansion

$$\psi = \sqrt[3]{1 + \sqrt[3]{1 + \sqrt[3]{1 + \sqrt[3]{1 + \sqrt[3]{1 + \cdots}}}}}.$$

An amusing account of ψ is given in [20], where it is referred to as the Plastic constant (to contrast against the Golden constant). See also [2.30].

The so-called **Tribonacci sequence** [22, 23]

$$h_0 = 0, \qquad h_1 = 0, \qquad h_2 = 1, \qquad h_n = h_{n-1} + h_{n-2} + h_{n-3} \quad \text{for } n \geq 3$$

has an analogous limiting ratio

$$\chi = \left(\tfrac{19}{27} + \tfrac{\sqrt{33}}{9}\right)^{\frac{1}{3}} + \tfrac{4}{9}\left(\tfrac{19}{27} + \tfrac{\sqrt{33}}{9}\right)^{-\frac{1}{3}} + \tfrac{1}{3} = \tfrac{4}{3}\cos\left(\tfrac{1}{3}\arccos\left(\tfrac{19}{8}\right)\right) + \tfrac{1}{3}$$
$$= 1.8392867552\ldots,$$

that is, the real solution of $\chi^3 - \chi^2 - \chi - 1 = 0$. See [1.2.3]. Consider also the **four-numbers game**: Start with a 4-vector (a, b, c, d) of nonnegative real numbers and determine the cyclic absolute differences $(|b - a|, |c - b|, |d - c|, |a - d|)$. Iterate indefinitely. Under most circumstances (e.g., if a, b, c, d are each positive integers), the process terminates with the zero 4-vector after only a finite number of steps. Is this always true? No. It is known [24] that $v = (1, \chi, \chi^2, \chi^3)$ is a counterexample, as well as any positive scalar multiple of v, or linear combination with the 4-vector $(1, 1, 1, 1)$. Also, $w = (\chi^3, \chi^2 + \chi, \chi^2, 0)$ is a counterexample, as well as any positive scalar multiple of w, or linear combination with the 4-vector $(1, 1, 1, 1)$. These encompass all the possible exceptions. Note that, starting with w, one obtains v after one step.

1.2.3 Generalized Continued Fractions

Recall from [1.1.1] that generalized continued fractions are constructed via the replacement rule

$$p \to p + \frac{1}{q}, \qquad q \to q + \frac{p}{q}$$

applied to each new term in a particular generation. In particular, if $p = q = 1$, the partial convergents are equal to ratios of successive terms of the Tribonacci sequence, and hence converge to χ. By way of contrast, the replacement rule [25, 26]

$$r \to r + \cfrac{1}{r + \cfrac{1}{r}}$$

is associated with a root of $x^3 - rx^2 - r = 0$. If $r = 1$, the limiting value is

$$\left(\frac{29}{54} + \frac{\sqrt{93}}{18}\right)^{\frac{1}{3}} + \frac{1}{9}\left(\frac{29}{54} + \frac{\sqrt{93}}{18}\right)^{-\frac{1}{3}} + \frac{1}{3} = \frac{2}{3}\cos\left(\frac{1}{3}\arccos\left(\frac{29}{2}\right)\right) + \frac{1}{3}$$
$$= 1.4655712318\ldots.$$

Other higher-order analogs of the Golden mean are offered in [27–29].

1.2.4 Random Fibonacci Sequences

Consider the sequence of random variables

$$x_0 = 1, \qquad x_1 = 1, \qquad x_n = \pm x_{n-1} \pm x_{n-2} \quad \text{for } n \geq 2,$$

where the signs are equiprobable and independent. Viswanath [30–32] proved the surprising result that

$$\lim_{n\to\infty} \sqrt[n]{|x_n|} = 1.13198824\ldots$$

with probability 1. Embree & Trefethen [33] proved that generalized random linear recurrences of the form

$$x_n = x_{n-1} \pm \beta x_{n-2}$$

decay exponentially with probability 1 if $0 < \beta < 0.70258\ldots$ and grow exponentially with probability 1 if $\beta > 0.70258\ldots$.

1.2.5 Fibonacci Factorials

We mention the asymptotic result $\prod_{k=1}^{n} f_k \sim c \cdot \varphi^{n(n+1)/2} \cdot 5^{-n/2}$ as $n \to \infty$, where [34, 35]

$$c = \prod_{n=1}^{\infty}\left(1 - \frac{(-1)^n}{\varphi^{2n}}\right) = 1.2267420107\ldots.$$

See the related expression in [5.14].

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