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Mathematical Constants

STEVEN R. FINCH



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Contents

Preface	<i>page</i> xvii
Notation	xix
1 Well-Known Constants	1
1.1 Pythagoras' Constant, $\sqrt{2}$	1
1.1.1 Generalized Continued Fractions	3
1.1.2 Radical Denestings	4
1.2 The Golden Mean, φ	5
1.2.1 Analysis of a Radical Expansion	8
1.2.2 Cubic Variations of the Golden Mean	8
1.2.3 Generalized Continued Fractions	9
1.2.4 Random Fibonacci Sequences	10
1.2.5 Fibonacci Factorials	10
1.3 The Natural Logarithmic Base, e	12
1.3.1 Analysis of a Limit	14
1.3.2 Continued Fractions	15
1.3.3 The Logarithm of Two	15
1.4 Archimedes' Constant, π	17
1.4.1 Infinite Series	20
1.4.2 Infinite Products	21
1.4.3 Definite Integrals	22
1.4.4 Continued Fractions	23
1.4.5 Infinite Radical	23
1.4.6 Elliptic Functions	24
1.4.7 Unexpected Appearances	24
1.5 Euler–Mascheroni Constant, γ	28
1.5.1 Series and Products	30
1.5.2 Integrals	31
1.5.3 Generalized Euler Constants	32
1.5.4 Gamma Function	33

1.6	Apéry's Constant, $\zeta(3)$	40
1.6.1	Bernoulli Numbers	41
1.6.2	The Riemann Hypothesis	41
1.6.3	Series	42
1.6.4	Products	45
1.6.5	Integrals	45
1.6.6	Continued Fractions	46
1.6.7	Stirling Cycle Numbers	47
1.6.8	Polylogarithms	47
1.7	Catalan's Constant, G	53
1.7.1	Euler Numbers	54
1.7.2	Series	55
1.7.3	Products	56
1.7.4	Integrals	56
1.7.5	Continued Fractions	57
1.7.6	Inverse Tangent Integral	57
1.8	Khintchine–Lévy Constants	59
1.8.1	Alternative Representations	61
1.8.2	Derived Constants	63
1.8.3	Complex Analog	63
1.9	Feigenbaum–Coullet–Tresser Constants	65
1.9.1	Generalized Feigenbaum Constants	68
1.9.2	Quadratic Planar Maps	69
1.9.3	Cvitanovic–Feigenbaum Functional Equation	69
1.9.4	Golden and Silver Circle Maps	71
1.10	Madelung's Constant	76
1.10.1	Lattice Sums and Euler's Constant	78
1.11	Chaitin's Constant	81
2	Constants Associated with Number Theory	84
2.1	Hardy–Littlewood Constants	84
2.1.1	Primes Represented by Quadratics	87
2.1.2	Goldbach's Conjecture	87
2.1.3	Primes Represented by Cubics	89
2.2	Meissel–Mertens Constants	94
2.2.1	Quadratic Residues	96
2.3	Landau–Ramanujan Constant	98
2.3.1	Variations	99
2.4	Artin's Constant	104
2.4.1	Relatives	106
2.4.2	Correction Factors	107
2.5	Hafner–Sarnak–McCurley Constant	110
2.5.1	Carefree Couples	110

2.6	Niven's Constant	112
2.6.1	Square-Full and Cube-Full Integers	113
2.7	Euler Totient Constants	115
2.8	Pell–Stevenhagen Constants	119
2.9	Alladi–Grinstead Constant	120
2.10	Sierpinski's Constant	122
2.10.1	Circle and Divisor Problems	123
2.11	Abundant Numbers Density Constant	126
2.12	Linnik's Constant	127
2.13	Mills' Constant	130
2.14	Brun's Constant	133
2.15	Glaisher–Kinkelin Constant	135
2.15.1	Generalized Glaisher Constants	136
2.15.2	Multiple Barnes Functions	137
2.15.3	GUE Hypothesis	138
2.16	Stolarsky–Harborth Constant	145
2.16.1	Digital Sums	146
2.16.2	Ulam 1-Additive Sequences	147
2.16.3	Alternating Bit Sets	148
2.17	Gauss–Kuzmin–Wirsing Constant	151
2.17.1	Ruelle-Mayer Operators	152
2.17.2	Asymptotic Normality	154
2.17.3	Bounded Partial Denominators	154
2.18	Porter–Hensley Constants	156
2.18.1	Binary Euclidean Algorithm	158
2.18.2	Worst-Case Analysis	159
2.19	Vallée's Constant	160
2.19.1	Continuant Polynomials	162
2.20	Erdős' Reciprocal Sum Constants	163
2.20.1	A -Sequences	163
2.20.2	B_2 -Sequences	164
2.20.3	Nonaveraging Sequences	164
2.21	Stieltjes Constants	166
2.21.1	Generalized Gamma Functions	169
2.22	Liouville–Roth Constants	171
2.23	Diophantine Approximation Constants	174
2.24	Self-Numbers Density Constant	179
2.25	Cameron's Sum-Free Set Constants	180
2.26	Triple-Free Set Constants	183
2.27	Erdős–Lebensold Constant	185
2.27.1	Finite Case	185
2.27.2	Infinite Case	186
2.27.3	Generalizations	187

2.28	Erdős' Sum-Distinct Set Constant	188
2.29	Fast Matrix Multiplication Constants	191
2.30	Pisot-Vijayaraghavan-Salem Constants	192
	2.30.1 Powers of $3/2$ Modulo One	194
2.31	Freiman's Constant	199
	2.31.1 Lagrange Spectrum	199
	2.31.2 Markov Spectrum	199
	2.31.3 Markov-Hurwitz Equation	200
	2.31.4 Hall's Ray	201
	2.31.5 L and M Compared	202
2.32	De Bruijn-Newman Constant	203
2.33	Hall-Montgomery Constant	205
3	Constants Associated with Analytic Inequalities	208
3.1	Shapiro-Drinfeld Constant	208
	3.1.1 Djokovic's Conjecture	210
3.2	Carlson-Levin Constants	211
3.3	Landau-Kolmogorov Constants	212
	3.3.1 $L_\infty(0, \infty)$ Case	212
	3.3.2 $L_\infty(-\infty, \infty)$ Case	213
	3.3.3 $L_2(-\infty, \infty)$ Case	213
	3.3.4 $L_2(0, \infty)$ Case	214
3.4	Hilbert's Constants	216
3.5	Copson-de Bruijn Constant	217
3.6	Sobolev Isoperimetric Constants	219
	3.6.1 String Inequality	220
	3.6.2 Rod Inequality	220
	3.6.3 Membrane Inequality	221
	3.6.4 Plate Inequality	222
	3.6.5 Other Variations	222
3.7	Korn Constants	225
3.8	Whitney-Mikhlin Extension Constants	227
3.9	Zolotarev-Schur Constant	229
	3.9.1 Sewell's Problem on an Ellipse	230
3.10	Kneser-Mahler Polynomial Constants	231
3.11	Grothendieck's Constants	235
3.12	Du Bois Reymond's Constants	237
3.13	Steinitz Constants	240
	3.13.1 Motivation	240
	3.13.2 Definitions	240
	3.13.3 Results	241
3.14	Young-Fejér-Jackson Constants	242
	3.14.1 Nonnegativity of Cosine Sums	242

3.14.2	Positivity of Sine Sums	243
3.14.3	Uniform Boundedness	243
3.15	Van der Corput's Constant	245
3.16	Turán's Power Sum Constants	246
4	Constants Associated with the Approximation of Functions	248
4.1	Gibbs–Wilbraham Constant	248
4.2	Lebesgue Constants	250
4.2.1	Trigonometric Fourier Series	250
4.2.2	Lagrange Interpolation	252
4.3	Achieser–Krein–Favard Constants	255
4.4	Bernstein's Constant	257
4.5	The “One-Ninth” Constant	259
4.6	Fransén–Robinson Constant	262
4.7	Berry–Esseen Constant	264
4.8	Laplace Limit Constant	266
4.9	Integer Chebyshev Constant	268
4.9.1	Transfinite Diameter	271
5	Constants Associated with Enumerating Discrete Structures	273
5.1	Abelian Group Enumeration Constants	274
5.1.1	Semisimple Associative Rings	277
5.2	Pythagorean Triple Constants	278
5.3	Rényi's Parking Constant	278
5.3.1	Random Sequential Adsorption	280
5.4	Golomb–Dickman Constant	284
5.4.1	Symmetric Group	287
5.4.2	Random Mapping Statistics	287
5.5	Kalmár's Composition Constant	292
5.6	Otter's Tree Enumeration Constants	295
5.6.1	Chemical Isomers	298
5.6.2	More Tree Varieties	301
5.6.3	Attributes	303
5.6.4	Forests	305
5.6.5	Cacti and 2-Trees	305
5.6.6	Mapping Patterns	307
5.6.7	More Graph Varieties	309
5.6.8	Data Structures	310
5.6.9	Galton–Watson Branching Process	312
5.6.10	Erdős–Rényi Evolutionary Process	312
5.7	Lengyel's Constant	316
5.7.1	Stirling Partition Numbers	316

5.7.2	Chains in the Subset Lattice of S	317
5.7.3	Chains in the Partition Lattice of S	318
5.7.4	Random Chains	320
5.8	Takeuchi–Prellberg Constant	321
5.9	Pólya’s Random Walk Constants	322
5.9.1	Intersections and Trappings	327
5.9.2	Holonomicity	328
5.10	Self-Avoiding Walk Constants	331
5.10.1	Polygons and Trails	333
5.10.2	Rook Paths on a Chessboard	334
5.10.3	Meanders and Stamp Foldings	334
5.11	Feller’s Coin Tossing Constants	339
5.12	Hard Square Entropy Constant	342
5.12.1	Phase Transitions in Lattice Gas Models	344
5.13	Binary Search Tree Constants	349
5.14	Digital Search Tree Constants	354
5.14.1	Other Connections	357
5.14.2	Approximate Counting	359
5.15	Optimal Stopping Constants	361
5.16	Extreme Value Constants	363
5.17	Pattern-Free Word Constants	367
5.18	Percolation Cluster Density Constants	371
5.18.1	Critical Probability	372
5.18.2	Series Expansions	373
5.18.3	Variations	374
5.19	Klarner’s Polyomino Constant	378
5.20	Longest Subsequence Constants	382
5.20.1	Increasing Subsequences	382
5.20.2	Common Subsequences	384
5.21	k -Satisfiability Constants	387
5.22	Lenz–Ising Constants	391
5.22.1	Low-Temperature Series Expansions	392
5.22.2	High-Temperature Series Expansions	393
5.22.3	Phase Transitions in Ferromagnetic Models	394
5.22.4	Critical Temperature	396
5.22.5	Magnetic Susceptibility	397
5.22.6	Q and P Moments	398
5.22.7	Painlevé III Equation	401
5.23	Monomer–Dimer Constants	406
5.23.1	2D Domino Tilings	406
5.23.2	Lozenges and Bibones	408
5.23.3	3D Domino Tilings	408
5.24	Lieb’s Square Ice Constant	412
5.24.1	Coloring	413

5.24.2	Folding	414
5.24.3	Atomic Arrangement in an Ice Crystal	415
5.25	Tutte–Beraha Constants	416
6	Constants Associated with Functional Iteration	420
6.1	Gauss’ Lemniscate Constant	420
6.1.1	Weierstrass Pe Function	422
6.2	Euler–Gompertz Constant	423
6.2.1	Exponential Integral	424
6.2.2	Logarithmic Integral	425
6.2.3	Divergent Series	425
6.2.4	Survival Analysis	425
6.3	Kepler–Bouwkamp Constant	428
6.4	Grossman’s Constant	429
6.5	Plouffe’s Constant	430
6.6	Lehmer’s Constant	433
6.7	Cahen’s Constant	434
6.8	Prouhet–Thue–Morse Constant	436
6.8.1	Probabilistic Counting	437
6.8.2	Non-Integer Bases	438
6.8.3	External Arguments	439
6.8.4	Fibonacci Word	439
6.8.5	Paper Folding	439
6.9	Minkowski–Bower Constant	441
6.10	Quadratic Recurrence Constants	443
6.11	Iterated Exponential Constants	448
6.11.1	Exponential Recurrences	450
6.12	Conway’s Constant	452
7	Constants Associated with Complex Analysis	456
7.1	Bloch–Landau Constants	456
7.2	Masser–Gramain Constant	459
7.3	Whittaker–Goncharov Constants	461
7.3.1	Goncharov Polynomials	463
7.3.2	Remainder Polynomials	464
7.4	John Constant	465
7.5	Hayman Constants	468
7.5.1	Hayman–Kjellberg	468
7.5.2	Hayman–Korenblum	468
7.5.3	Hayman–Stewart	469
7.5.4	Hayman–Wu	470
7.6	Littlewood–Clunie–Pommerenke Constants	470
7.6.1	Alpha	470

7.6.2	Beta and Gamma	471
7.6.3	Conjectural Relations	472
7.7	Riesz–Kolmogorov Constants	473
7.8	Grötzsch Ring Constants	475
7.8.1	Formula for $a(r)$	477
8	Constants Associated with Geometry	479
8.1	Geometric Probability Constants	479
8.2	Circular Coverage Constants	484
8.3	Universal Coverage Constants	489
8.3.1	Translation Covers	490
8.4	Moser’s Worm Constant	491
8.4.1	Broadest Curve of Unit Length	493
8.4.2	Closed Worms	493
8.4.3	Translation Covers	495
8.5	Traveling Salesman Constants	497
8.5.1	Random Links TSP	498
8.5.2	Minimum Spanning Trees	499
8.5.3	Minimum Matching	500
8.6	Steiner Tree Constants	503
8.7	Hermite’s Constants	506
8.8	Tammes’ Constants	508
8.9	Hyperbolic Volume Constants	511
8.10	Reuleaux Triangle Constants	513
8.11	Beam Detection Constant	515
8.12	Moving Sofa Constant	519
8.13	Calabi’s Triangle Constant	523
8.14	DeVicci’s Tesseract Constant	524
8.15	Graham’s Hexagon Constant	526
8.16	Heilbronn Triangle Constants	527
8.17	Keakeya–Besicovitch Constants	530
8.18	Rectilinear Crossing Constant	532
8.19	Circumradius–Inradius Constants	534
8.20	Apollonian Packing Constant	537
8.21	Rendezvous Constants	539
	Table of Constants	543
	Author Index	567
	Subject Index	593
	Added in Press	601

Well-Known Constants

1.1 Pythagoras' Constant, $\sqrt{2}$

The diagonal of a unit square has length $\sqrt{2} = 1.4142135623 \dots$. A theory, proposed by the Pythagorean school of philosophy, maintained that all geometric magnitudes could be expressed by rational numbers. The sides of a square were expected to be commensurable with its diagonals, in the sense that certain integer multiples of one would be equivalent to integer multiples of the other. This theory was shattered by the discovery that $\sqrt{2}$ is irrational [1–4].

Here are two proofs of the irrationality of $\sqrt{2}$, the first based on divisibility properties of the integers and the second using well ordering.

- If $\sqrt{2}$ were rational, then the equation $p^2 = 2q^2$ would be solvable in integers p and q , which are assumed to be in lowest terms. Since p^2 is even, p itself must be even and so has the form $p = 2r$. This leads to $2q^2 = 4r^2$ and thus q must also be even. But this contradicts the assumption that p and q were in lowest terms.
- If $\sqrt{2}$ were rational, then there would be a least positive integer s such that $s\sqrt{2}$ is an integer. Since $1 < 2$, it follows that $1 < \sqrt{2}$ and thus $t = s \cdot (\sqrt{2} - 1)$ is a positive integer. Also $t\sqrt{2} = s \cdot (\sqrt{2} - 1)\sqrt{2} = 2s - s\sqrt{2}$ is an integer and clearly $t < s$. But this contradicts the assumption that s was the smallest such integer.

Newton's method for solving equations gives rise to the following first-order recurrence, which is very fast and often implemented:

$$x_0 = 1, \quad x_k = \frac{x_{k-1}}{2} + \frac{1}{x_{k-1}} \quad \text{for } k \geq 1, \quad \lim_{k \rightarrow \infty} x_k = \sqrt{2}.$$

Another first-order recurrence [5] yields the reciprocal of $\sqrt{2}$:

$$y_0 = \frac{1}{2}, \quad y_k = y_{k-1} \left(\frac{3}{2} - y_{k-1}^2 \right) \quad \text{for } k \geq 1, \quad \lim_{k \rightarrow \infty} y_k = \frac{1}{\sqrt{2}}.$$

The binomial series, also due to Newton, provides two interesting summations [6]:

$$1 + \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{2^{2n}(2n-1)} \binom{2n}{n} = 1 + \frac{1}{2} - \frac{1}{2 \cdot 4} + \frac{1 \cdot 3}{2 \cdot 4 \cdot 6} - \dots = \sqrt{2},$$

$$1 + \sum_{n=1}^{\infty} \frac{(-1)^n}{2^{2n}} \binom{2n}{n} = 1 - \frac{1}{2} + \frac{1 \cdot 3}{2 \cdot 4} - \frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6} + \dots = \frac{1}{\sqrt{2}}.$$

The latter is extended in [1.5.4]. We mention two beautiful infinite products [5, 7, 8]

$$\prod_{n=1}^{\infty} \left(1 + \frac{(-1)^{n+1}}{2n-1}\right) = \left(1 + \frac{1}{1}\right) \left(1 - \frac{1}{3}\right) \left(1 + \frac{1}{5}\right) \left(1 - \frac{1}{7}\right) \dots = \sqrt{2},$$

$$\prod_{n=1}^{\infty} \left(1 - \frac{1}{4(2n-1)^2}\right) = \frac{1 \cdot 3}{2 \cdot 2} \cdot \frac{5 \cdot 7}{6 \cdot 6} \cdot \frac{9 \cdot 11}{10 \cdot 10} \cdot \frac{13 \cdot 15}{14 \cdot 14} \dots = \frac{1}{\sqrt{2}}$$

and the regular continued fraction [9]

$$2 + \frac{1}{2 + \frac{1}{2 + \frac{1}{2 + \dots}}} = 2 + \frac{1}{|2} + \frac{1}{|2} + \frac{1}{|2} + \dots = 1 + \sqrt{2} = (-1 + \sqrt{2})^{-1},$$

which is related to **Pell's sequence**

$$a_0 = 0, \quad a_1 = 1, \quad a_n = 2a_{n-1} + a_{n-2} \quad \text{for } n \geq 2$$

via the limiting formula

$$\lim_{n \rightarrow \infty} \frac{a_{n+1}}{a_n} = 1 + \sqrt{2}.$$

This is completely analogous to the famous connection between the Golden mean φ and Fibonacci's sequence [1.2]. See also Figure 1.1.

Viète's remarkable product for Archimedes' constant π [1.4.2] involves only the number 2 and repeated square-root extractions. Another expression connecting π and radicals appears in [1.4.5].

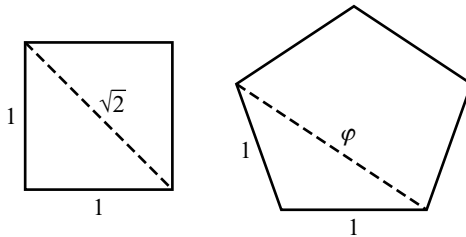


Figure 1.1. The diagonal of a regular unit pentagon, connecting any two nonadjacent corners, has length given by the Golden mean φ (rather than by Pythagoras' constant).

We return finally to irrationality issues: There obviously exist rationals x and y such that x^y is irrational (just take $x = 2$ and $y = 1/2$). Do there exist *irrationals* x and y such that x^y is *rational*? The answer to this is very striking. Let

$$z = \sqrt{2}^{\sqrt{2}}.$$

If z is rational, then take $x = y = \sqrt{2}$. If z is irrational, then take $x = z$ and $y = \sqrt{2}$, and clearly $x^y = 2$. Thus we have answered the question (“yes”) without addressing the actual arithmetical nature of z . In fact, z is transcendental by the Gel'fond–Schneider theorem [10], proved in 1934, and hence is irrational. There are many unsolved problems in this area of mathematics; for example, we do not know whether

$$\sqrt{2}^z = \sqrt{2}^{\sqrt{2}^{\sqrt{2}}}$$

is irrational (let alone transcendental).

1.1.1 Generalized Continued Fractions

It is well known that any quadratic irrational possesses a periodic regular continued fraction expansion and vice versa. Comparatively few people have examined the generalized continued fraction [11–17]

$$w(p, q) = q + \frac{1}{q + \frac{p + \dots}{q + \dots}},$$

$$q + \frac{p + \frac{1}{q + \dots}}{q + \frac{p + \dots}{q + \dots}},$$

which exhibits a fractal-like construction. Each *new* term in a particular generation (i.e., in a partial convergent) is replaced according to the rules

$$p \rightarrow p + \frac{1}{q}, \quad q \rightarrow q + \frac{p}{q}$$

in the next generation. Clearly

$$w = q + \frac{p + \frac{1}{w}}{w}; \quad \text{that is, } w^3 - qw^2 - pw - 1 = 0.$$

In the special case $p = q = 3$, the higher-order continued fraction converges to $(-1 + \sqrt[3]{2})^{-1}$. It is conjectured that regular continued fractions for cubic irrationals behave like those for almost all real numbers [18–21], and no patterns are evident. The ordinary replacement rule

$$r \rightarrow r + \frac{1}{r}$$

is sufficient for the study of quadratic irrationals, but requires extension for broader classes of algebraic numbers.

Two alternative representations of $\sqrt[3]{2}$ are as follows [22]:

$$\sqrt[3]{2} = 1 + \frac{1}{3 + \frac{1}{a + \frac{1}{b}}}, \quad \text{where } a = 3 + \frac{3}{a} + \frac{1}{b}, \quad b = 12 + \frac{10}{a} + \frac{3}{b}$$

and [23]

$$\sqrt[3]{2} = 1 + \frac{1|}{|3} + \frac{2|}{|2} + \frac{4|}{|9} + \frac{5|}{|2} + \frac{7|}{|15} + \frac{8|}{|2} + \frac{10|}{|21} + \frac{11|}{|2} + \dots$$

Other usages of the phrase “generalized continued fractions” include those in [24], with application to simultaneous Diophantine approximation, and in [25], with a geometric interpretation involving the boundaries of convex hulls.

1.1.2 Radical Denestings

We mention two striking radical denestings due to Ramanujan:

$$\sqrt[3]{\sqrt[3]{2} - 1} = \sqrt[3]{\frac{1}{9}} - \sqrt[3]{\frac{2}{9}} + \sqrt[3]{\frac{4}{9}}, \quad \sqrt[2]{\sqrt[3]{5} - \sqrt[3]{4}} = \frac{1}{3} \left(\sqrt[3]{2} + \sqrt[3]{20} - \sqrt[3]{25} \right).$$

Such simplifications are an important part of computer algebra systems [26].

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1.2 The Golden Mean, φ

Consider a line segment:



What is the most “pleasing” division of this line segment into two parts? Some people might say at the halfway point:



Others might say at the one-quarter or three-quarters point. The “correct answer” is, however, none of these, and is supposedly found in Western art from the ancient Greeks onward (aestheticians speak of it as the principle of “dynamic symmetry”):



If the right-hand portion is of length $v = 1$, then the left-hand portion is of length $u = 1.618\dots$. A line segment partitioned as such is said to be divided in Golden or Divine section. What is the justification for endowing this particular division with such elevated status? The length u , as drawn, is to the whole length $u + v$, as the length v is to u :

$$\frac{u}{u+v} = \frac{v}{u}.$$

Letting $\varphi = u/v$, solve for φ via the observation that

$$1 + \frac{1}{\varphi} = 1 + \frac{v}{u} = \frac{u+v}{u} = \frac{u}{v} = \varphi.$$

The positive root of the resulting quadratic equation $\varphi^2 - \varphi - 1 = 0$ is

$$\varphi = \frac{1 + \sqrt{5}}{2} = 1.6180339887\dots,$$

which is called the **Golden mean** or **Divine proportion** [1, 2].

The constant φ is intricately related to **Fibonacci's sequence**

$$f_0 = 0, \quad f_1 = 1, \quad f_n = f_{n-1} + f_{n-2} \quad \text{for } n \geq 2.$$

This sequence models (in a naive way) the growth of a rabbit population. Rabbits are assumed to start having bunnies once a month after they are two months old; they always give birth to twins (one male bunny and one female bunny), they never die, and they never stop propagating. The number of rabbit pairs after n months is f_n .

What can φ possibly have in common with $\{f_n\}$? This is one of the most remarkable ideas in all of mathematics. The partial convergents leading up to the regular continued fraction representation of φ ,

$$\varphi = 1 + \frac{1}{1 + \frac{1}{1 + \frac{1}{1 + \dots}}} = 1 + \frac{1}{1} + \frac{1}{1} + \frac{1}{1} + \dots,$$

are all ratios of successive Fibonacci numbers; hence

$$\lim_{n \rightarrow \infty} \frac{f_{n+1}}{f_n} = \varphi.$$

This result is also true for arbitrary sequences satisfying the same recursion $f_n = f_{n-1} + f_{n-2}$, assuming that the initial terms f_0 and f_1 are distinct [3, 4].

The rich geometric connection between the Golden mean and Fibonacci's sequence is seen in Figure 1.2. Starting with a single Golden rectangle (of length φ and width 1), there is a natural sequence of nested Golden rectangles obtained by removing the leftmost square from the first rectangle, the topmost square from the second rectangle, etc. The length and width of the n^{th} Golden rectangle can be written as linear expressions $a + b\varphi$, where the coefficients a and b are always Fibonacci numbers. These Golden rectangles can be inscribed in a logarithmic spiral as pictured. Assume that the lower left corner of the first rectangle is the origin of an xy -coordinate system.

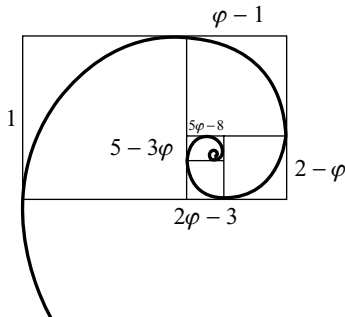


Figure 1.2. The Golden spiral circumscribes the sequence of Golden rectangles.

The accumulation point for the spiral can be proved to be $(\frac{1}{5}(1 + 3\varphi), \frac{1}{5}(3 - \varphi))$. Such logarithmic spirals are “equiangular” in the sense that every line through (x_∞, y_∞) cuts across the spiral at a constant angle ξ . In this way, logarithmic spirals generalize ordinary circles (for which $\xi = 90^\circ$). The logarithmic spiral pictured gives rise to the constant angle $\xi = \operatorname{arccot}(\frac{2}{\pi} \ln(\varphi)) = 72.968\dots^\circ$. Logarithmic spirals are evidently found throughout nature; for example, the shell of a chambered nautilus, the tusks of an elephant, and patterns in sunflowers and pine cones [4–6].

Another geometric application of the Golden mean arises when inscribing a regular pentagon within a given circle by ruler and compass. This is related to the fact that

$$2 \cos\left(\frac{\pi}{5}\right) = \varphi, \quad 2 \sin\left(\frac{\pi}{5}\right) = \sqrt{3 - \varphi}.$$

The Golden mean, just as it has a simple regular continued fraction expansion, also has a simple radical expansion [7]

$$\varphi = \sqrt{1 + \sqrt{1 + \sqrt{1 + \sqrt{1 + \sqrt{1 + \dots}}}}}$$

The manner in which this expansion converges to φ is discussed in [1.2.1]. Like Pythagoras’ constant [1.1], the Golden mean is irrational and simple proofs are given in [8, 9].

Here is a series [10] involving φ :

$$\begin{aligned} \frac{2\sqrt{5}}{5} \ln(\varphi) &= \left(1 - \frac{1}{2} - \frac{1}{3} + \frac{1}{4}\right) + \left(\frac{1}{6} - \frac{1}{7} - \frac{1}{8} + \frac{1}{9}\right) \\ &+ \left(\frac{1}{11} - \frac{1}{12} - \frac{1}{13} + \frac{1}{14}\right) + \dots, \end{aligned}$$

which reminds us of certain series connected with Archimedes’ constant [1.4.1]. A direct expression for φ as a sum can be obtained from the Taylor series for the square root function, expanded about 4. The Fibonacci numbers appear in yet another representation [11] of φ :

$$4 - \varphi = \sum_{n=0}^{\infty} \frac{1}{f_{2^n}} = \frac{1}{f_1} + \frac{1}{f_2} + \frac{1}{f_4} + \frac{1}{f_8} + \dots$$

Among many other possible formulas involving φ , we mention the four Rogers–Ramanujan continued fractions

$$\begin{aligned} \frac{1}{\alpha - \varphi} \exp\left(-\frac{2\pi}{5}\right) &= 1 + \frac{e^{-2\pi}}{|1|} + \frac{e^{-4\pi}}{|1|} + \frac{e^{-6\pi}}{|1|} + \frac{e^{-8\pi}}{|1|} + \dots, \\ \frac{1}{\beta - \varphi} \exp\left(-\frac{2\pi}{\sqrt{5}}\right) &= 1 + \frac{e^{-2\pi\sqrt{5}}}{|1|} + \frac{e^{-4\pi\sqrt{5}}}{|1|} + \frac{e^{-6\pi\sqrt{5}}}{|1|} + \frac{e^{-8\pi\sqrt{5}}}{|1|} + \dots, \\ \frac{1}{\kappa - (\varphi - 1)} \exp\left(-\frac{\pi}{5}\right) &= 1 - \frac{e^{-\pi}}{|1|} + \frac{e^{-2\pi}}{|1|} - \frac{e^{-3\pi}}{|1|} + \frac{e^{-4\pi}}{|1|} - + \dots, \\ \frac{1}{\lambda - (\varphi - 1)} \exp\left(-\frac{\pi}{\sqrt{5}}\right) &= 1 - \frac{e^{-\pi\sqrt{5}}}{|1|} + \frac{e^{-2\pi\sqrt{5}}}{|1|} - \frac{e^{-3\pi\sqrt{5}}}{|1|} + \frac{e^{-4\pi\sqrt{5}}}{|1|} - + \dots, \end{aligned}$$

where

$$\alpha = (\varphi\sqrt{5})^{\frac{1}{2}}, \quad \alpha' = \frac{1}{\sqrt{5}} \left((\varphi - 1)\sqrt{5} \right)^{\frac{5}{2}}, \quad \beta = \frac{\sqrt{5}}{1 + \sqrt[5]{\alpha' - 1}},$$

$$\kappa = \left((\varphi - 1)\sqrt{5} \right)^{\frac{1}{2}}, \quad \kappa' = \frac{1}{\sqrt{5}} \left(\varphi\sqrt{5} \right)^{\frac{5}{2}}, \quad \lambda = \frac{\sqrt{5}}{1 + \sqrt[5]{\kappa' - 1}}.$$

The fourth evaluation is due to Ramanathan [9, 12–16].

1.2.1 Analysis of a Radical Expansion

The radical expansion [1.2] for φ can be rewritten as a sequence $\{\varphi_n\}$:

$$\varphi_1 = 1, \quad \varphi_n = \sqrt{1 + \varphi_{n-1}} \quad \text{for } n \geq 2.$$

Paris [17] proved that the rate in which φ_n approaches the limit φ is given by

$$\varphi - \varphi_n \sim \frac{2C}{(2\varphi)^n} \quad \text{as } n \rightarrow \infty,$$

where $C = 1.0986419643\dots$ is a new constant. Here is an exact characterization of C . Let $F(x)$ be the analytic solution of the functional equation

$$F(x) = 2\varphi F(\varphi - \sqrt{\varphi^2 - x}), \quad |x| < \varphi^2,$$

subject to the initial conditions $F(0) = 0$ and $F'(0) = 1$. Then $C = \varphi F(1/\varphi)$. A power-series technique can be used to evaluate C numerically from these formulas. It is simpler, however, to use the following product:

$$C = \prod_{n=2}^{\infty} \frac{2\varphi}{\varphi + \varphi_n},$$

which is stable and converges quickly [18].

Another interesting constant is defined via the radical expression [7, 19]

$$\sqrt{1 + \sqrt{2 + \sqrt{3 + \sqrt{4 + \sqrt{5 + \dots}}}}} = 1.7579327566\dots,$$

but no expression of this in terms of other constants is known.

1.2.2 Cubic Variations of the Golden Mean

Perrin's sequence is defined by

$$g_0 = 3, \quad g_1 = 0, \quad g_2 = 2, \quad g_n = g_{n-2} + g_{n-3} \quad \text{for } n \geq 3$$

and has the property that $n > 1$ divides g_n if n is prime [20, 21]. The limit of ratios of successive Perrin numbers

$$\psi = \lim_{n \rightarrow \infty} \frac{g_{n+1}}{g_n}$$

satisfies $\psi^3 - \psi - 1 = 0$ and is given by

$$\begin{aligned}\psi &= \left(\frac{1}{2} + \frac{\sqrt{69}}{18}\right)^{\frac{1}{3}} + \frac{1}{3} \left(\frac{1}{2} + \frac{\sqrt{69}}{18}\right)^{-\frac{1}{3}} = \frac{2\sqrt{3}}{3} \cos\left(\frac{1}{3} \arccos\left(\frac{3\sqrt{3}}{2}\right)\right) \\ &= 1.3247179572\dots\end{aligned}$$

This also has the radical expansion

$$\psi = \sqrt[3]{1 + \sqrt[3]{1 + \sqrt[3]{1 + \sqrt[3]{1 + \sqrt[3]{1 + \dots}}}}}$$

An amusing account of ψ is given in [20], where it is referred to as the Plastic constant (to contrast against the Golden constant). See also [2.30].

The so-called **Tribonacci sequence** [22, 23]

$$h_0 = 0, \quad h_1 = 0, \quad h_2 = 1, \quad h_n = h_{n-1} + h_{n-2} + h_{n-3} \quad \text{for } n \geq 3$$

has an analogous limiting ratio

$$\begin{aligned}\chi &= \left(\frac{19}{27} + \frac{\sqrt{33}}{9}\right)^{\frac{1}{3}} + \frac{4}{9} \left(\frac{19}{27} + \frac{\sqrt{33}}{9}\right)^{-\frac{1}{3}} + \frac{1}{3} = \frac{4}{3} \cos\left(\frac{1}{3} \arccos\left(\frac{19}{8}\right)\right) + \frac{1}{3} \\ &= 1.8392867552\dots,\end{aligned}$$

that is, the real solution of $\chi^3 - \chi^2 - \chi - 1 = 0$. See [1.2.3]. Consider also the **four-numbers game**: Start with a 4-vector (a, b, c, d) of nonnegative real numbers and determine the cyclic absolute differences $(|b - a|, |c - b|, |d - c|, |a - d|)$. Iterate indefinitely. Under most circumstances (e.g., if a, b, c, d are each positive integers), the process terminates with the zero 4-vector after only a finite number of steps. Is this always true? No. It is known [24] that $v = (1, \chi, \chi^2, \chi^3)$ is a counterexample, as well as any positive scalar multiple of v , or linear combination with the 4-vector $(1, 1, 1, 1)$. Also, $w = (\chi^3, \chi^2 + \chi, \chi^2, 0)$ is a counterexample, as well as any positive scalar multiple of w , or linear combination with the 4-vector $(1, 1, 1, 1)$. These encompass all the possible exceptions. Note that, starting with w , one obtains v after one step.

1.2.3 Generalized Continued Fractions

Recall from [1.1.1] that generalized continued fractions are constructed via the replacement rule

$$p \rightarrow p + \frac{1}{q}, \quad q \rightarrow q + \frac{p}{q}$$

applied to each new term in a particular generation. In particular, if $p = q = 1$, the partial convergents are equal to ratios of successive terms of the Tribonacci sequence, and hence converge to χ . By way of contrast, the replacement rule [25, 26]

$$r \rightarrow r + \frac{1}{r + \frac{1}{r}}$$

is associated with a root of $x^3 - rx^2 - r = 0$. If $r = 1$, the limiting value is

$$\begin{aligned} \left(\frac{29}{54} + \frac{\sqrt{93}}{18}\right)^{\frac{1}{3}} + \frac{1}{9} \left(\frac{29}{54} + \frac{\sqrt{93}}{18}\right)^{-\frac{1}{3}} + \frac{1}{3} &= \frac{2}{3} \cos\left(\frac{1}{3} \arccos\left(\frac{29}{2}\right)\right) + \frac{1}{3} \\ &= 1.4655712318\dots \end{aligned}$$

Other higher-order analogs of the Golden mean are offered in [27–29].

1.2.4 Random Fibonacci Sequences

Consider the sequence of random variables

$$x_0 = 1, \quad x_1 = 1, \quad x_n = \pm x_{n-1} \pm x_{n-2} \quad \text{for } n \geq 2,$$

where the signs are equiprobable and independent. Viswanath [30–32] proved the surprising result that

$$\lim_{n \rightarrow \infty} \sqrt[n]{|x_n|} = 1.13198824\dots$$

with probability 1. Embree & Trefethen [33] proved that generalized random linear recurrences of the form

$$x_n = x_{n-1} \pm \beta x_{n-2}$$

decay exponentially with probability 1 if $0 < \beta < 0.70258\dots$ and grow exponentially with probability 1 if $\beta > 0.70258\dots$

1.2.5 Fibonacci Factorials

We mention the asymptotic result $\prod_{k=1}^n f_k \sim c \cdot \varphi^{n(n+1)/2} \cdot 5^{-n/2}$ as $n \rightarrow \infty$, where [34, 35]

$$c = \prod_{n=1}^{\infty} \left(1 - \frac{(-1)^n}{\varphi^{2n}}\right) = 1.2267420107\dots$$

See the related expression in [5.14].

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