The Emergence of Nonlinear Programming: Interactions between Practical Mathematics and Mathematics Proper

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The beginning of the modern mathematical theory of nonlinear programming can be dated back to 1950—quite precisely to the Second Berkeley Symposium on Mathematical Statistics and Probability held in Berkeley, California. At this meeting Albert W. Tucker, a mathematician from Princeton, presented a paper with the title Nonlinear Programming which he had written together with another Princeton mathematician, Harold W. Kuhn. After the meeting the paper was published in the proceedings of the Symposium and for the first time the name *nonlinear programming* appeared in the literature [Kuhn and Tucker, 1950].

In the paper Kuhn and Tucker defined the nonlinear programming problem—or *maximum problem* as they called it—as follows:

To find an x^0 that maximizes $g(x)$ constrained by $Fx \geqq 0$, $x \geqq 0$.

Here Fx is an m-vector $(f_1(x), \ldots, f_m(x))$, where $f_1(x), \ldots, f_m(x)$ are differentiable functions of x defined for $x \geqq 0$, and $g(x)$ is a differentiable function of x also defined for $x \geqq 0$. In words, a nonlinear programming problem is a finite-dimensional optimization problem subject to *inequality* constraints.

Beyond introducing the nonlinear programming problem, they also proved the main theorem of the theory—the *Kuhn-Tucker theorem* which later became so famous. This theorem gives necessary conditions for the existence of an optimal solution to a nonlinear programming problem, and launched the theory of nonlinear programming.

On the first page of the paper Kuhn and Tucker also revealed their sponsor:

This work was done under contracts with the Office of Naval Research.

In fact the joint work of Kuhn and Tucker on nonlinear programming grew out of an Office of Naval Research (ONR) project on game theory and linear programming which Tucker had led since 1948. The ONR was very important in the emergence and further development of nonlinear programming. It functioned as a bridge between practical problem-solving in the "real world" and university-based research in mathematics proper.

In what follows I will trace this interaction.

Mobilization of Scientists in the USA during World War II

Kuhn and Tucker's work on nonlinear programming took place within a project—originally on linear programming and game theory—which was initiated and financed by the Office of Naval Research. The background for understanding the interest of the Navy in such a project is the mobilization of civilian scientists in the USA during the second world war.

The financial structure of science changed radically in the USA during World War II. Under the leadership of the MIT electrical engineer Vannevar Bush, science statesmen like James B. Conant from Harvard University, Karl T. Compton from MIT, and Frank Jewett from AT&T's Bell Laboratories initiated and organized the mobilization of civilian scientists for the war effort. Before the war basic science had been financed largely by huge private (often family) foundations such as the Rockefeller Foundation. There was a widespread scepticism towards government influence on university science. There was a fear that government money in the universities would mean government control over scientific research. All this changed as a consequence of the war.[1]

Bush's vision was to create a system that could make scientific research in

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[1]See e.g., [Dupree, 1986] and [Zachary, 1997].

war methods, weapon development, and defense systems more efficient than they were before the war. He wanted to create an organization of civilian scientists who worked on these problems independent of the military. He established a system where the civilian scientists worked on military-related problems and research, bound by contracts not directly with the different military establishments but through the Office of Scientific Research and Development (OSRD), an organisation that came into being in May 1941. OSRD was directly under the Congress and as such it was independent of the military. Never before had civilians had so much influence on military affairs.

The system worked as follows: A scientist who was believed to be able to handle a desired project was contacted by OSRD. A contract was set up with the scientist as principal investigator. It was then up to the principal investigator to appoint additional staff to work on the project, typically his or her own graduate students. Most of the scientists did not relocate their work to military laboratories, instead they stayed where they were—at the universities and in industry.

This way of organizing the scientific war effort proved very efficient. One of the really great successes was the development of radar technology, which took place at MIT. It created of course also a lot of problems. There was rivalry between the armed forces, and not everybody in the military thought it was a good idea to have civilian scientists working on war-related issues outside the control of the military. There were, for example, a lot of problems connected with the use of radar, which the Navy initially simply refused to have anything to do with.[2]

Practical Mathematics: Solving A Logistic Problem

Not all the scientists who participated in the war were organized by OSRD. Some of the military establishments also had their own scientific staff. In 1941 the U.S. Air Force hired the mathematician George B. Dantzig, who worked on the so-called *programming planning methods*—a tool in the Air Force for handling huge logistic planning. An *Air Force Program* was a proposed schedule for activities. Dantzig gave the following explanation of such a program in 1951:

> *The levels of various activities such as training, maintenance, supply, and combat had to be adjusted in such a manner as not to exceed the availability of various equipment items. Indeed, activities should be so carefully phased that the necessary amounts of these various equipment items were available when they were supposed to be available, so that the activity could take place.* [Dantzig, 1951, p. 18]

According to Dantzig these methods for planning programmes were slow, expensive, and ineffective; they were built on personal experience, and incorporated a lot of *ad hoc* ground rules issued by those in charge. It took more than seven months to set up a program. Dantzig's job during the war was to train members of the Air Force staff to compute Air Force programs [Dantzig, 1968, p. 4].[3]

After the war Dantzig returned to the Air Force, where, from 1946 until 1952, he functioned as mathematical advisor for the Headquarters Staff of the U.S.A.F. The assignment he was hired to work on was to

> . . . *develop some kind of analog device which would accept, as input, equations of all types, basic data, and ground rules, and use these to generate as output a consistent Air Force plan.* [Dantzig, 1988, p. 12]

Around this time rumors about the computer started to circulate. This had a very profound influence on Dantzig's work. The idea of an "analog device" was rejected. Instead the work took a turn towards the development of what is now called linear programming. In the spring of 1947 the project SCOOP (Scientific Computation Of Optimum Programs) was established. The purpose of this project was twofold: to build a mathematical model for the programming problem, and to assist the development and construction of computers.[4]

It initiated a very intensive working period which resulted in a model that was reflected in the following mathematical problem:

> . . . *the minimization of a linear form subject to linear equations and inequalities.* [Dantzig, 1982, p. 44]

This is nowadays known as a linear programming problem; originally Dantzig called it *Programming in a Linear Structure.*

From Problem Solving to Theory: Linear Programming and Duality

The next problem was how to solve this model. Dantzig was advised to seek help from von Neumann [Dantzig, 1988, p. 13]. During the war von Neumann held a lot of advisory and consulting jobs in the military, and several of these continued after the war, so it was very natural for a mathematician working on a mathematical problem in the Air Force to pay a visit to him.[5]

It turned out to be a very fruitful meeting. Von Neumann had recently completed a book on game theory with Oskar Morgenstern, and—according to Dantzig—von Neumann immediately recognized the relationship between a linear programming problem and a two-person zero-sum game.[6]

The most important thing here for the development of nonlinear programming is that von Neumann provided Dantzig's *problem*—the Air Force problem—with a *theory,* game theory, which had a mathematical

[2] See [Zachary, 1997].

[3] For further readings on the origin of linear programming see [Dantzig, 1963, 1982, 1988, 1991] and [Dorfman, 1984].

[4] See [Brentjes, p. 177].

[5] In [Ulam, 1958, p. 42] there is an incomplete list of von Neumann's relationship with the military establishment.

[6] For a history of von Neumann's perception of the minimax theorem in two-person zero-sum games and the different mathematical contexts it gradually appeared in, see [Kjeldsen, 1997, 1999a, 1999b].

foundation in the theory of convexity and linear inequalities. This was very important for the later development of nonlinear programming, because it broadened the subject of linear programming and made it a subject for mathematical research. This widened the interest from a narrow militarily-defined problem to research areas within mathematics [Kjeldsen, 1999b].

The Significance of ONR: A Project in Game Theory and Linear Programming

The Office of Naval Research (ONR), which was established by the Navy in 1946, was an after-effect of the mobilization of scientists during World War II. Bush's organization during the war—OSRD—was an emergency organization, and it had been clear right from the beginning that the OSRD would disappear when the war ended. There was a common concern that the scientists would go back to their university duties after the war. There also was a strong belief that the US had to be strong scientifically in order to be strong militarily. A lot of people were concerned about the further financing of science after the war, military-related science as well as basic science.[7]

The Office of Naval Research was created to fill the void left by the disappearance of the OSRD. The ONR was organized after the model Bush created for the OSRD. The scientists continued to work in universities and industry. Their relationship with the ONR was based on contracts. Every project had a principal investigator, and the financial support from the ONR covered salaries during the summers, salaries for research assistants working on the projects, conferences, guests, etc. In this way the ONR functioned as a bridge between the interest of the military and peacetime research at the universities. During the first four years of its existence, the ONR was the main sponsor for government-supported research in the USA.[8]

The possible applications of Dantzig's model in connection with the development of the computer caused the ONR to set up a special logistic branch within its mathematics program. Mina Rees, who was the head of the Mathematics Division of the ONR, has described it like this:

> *. . . when, in the late 1940's the staff of our office became aware that some mathematical results obtained by George Dantzig, who was then working for the Air Force, could be used by the Navy to reduce the burdensome costs of their logistics operations, the possibilities were pointed out to the Deputy Chief of Naval Operations for Logistics. His enthusiasm for the possibilities presented by these results was so great that he called together all those senior officers who had anything to do with logistics, as well as their civilian counterparts, to hear what we always referred to as a "presentation". The outcome of this meeting was the establishment in the Office of Naval Research of a separate Logistics Branch with a separate research program. This has proved to be a most successful activity of the Mathematics Division of ONR, both in its usefulness to the Navy, and in its impact on industry and the universities.* [Rees, 1977a, p. 111]

The theoretical connection between the linear programming model and mathematics proper—the theory of games in applied mathematics and the theory of convexity and linear inequalities in pure mathematics—made it an obvious subject for a university-based ONR project.

Thus in the spring of 1948 Dantzig went back to Princeton, this time on behalf of the ONR, to discuss with John von Neumann the possibilities for a university-based project financed by ONR on linear programming, its relations to game theory, and the underlying mathematical structure.

In an interview Tucker has described how at this occasion he was introduced to Dantzig and gave him a ride to the train station. During this short car trip Dantzig gave Tucker a brief introduction to the linear programming problem. Tucker made a remark about a possible connection to Kirchoff-Maxwell's law of electric networks, and because of this remark Tucker was contacted by the ONR a few days later and asked if he would set up such a mathematics project [Interview, Albers and Alexanderson, 1985, p. 342–343].

Tucker agreed to become the principal investigator for the project, and that changed his research direction completely—until this moment he had been absorbed in research in topology. The same happened for Kuhn, who at the time was finishing a Ph.D. project on group theory. Kuhn went to Tucker to ask for summer employment in the summer of 1948, because he needed the additional income [Kuhn, 1998, Interview]. Tucker hired him, together with another graduate student, David Gale, to work with him on the ONR project.

What ONR did here was to initiate research in the connection between game theory and linear programming and the underlying mathematics. They placed this research in a university context and staffed it with mathematicians normally engaged in research in pure mathematics.

Tucker and his group presented the results of their work within the project at the first conference on linear programming which took place in Chicago in June 1949 [Koopmans, 1951]. They had developed the mathematical theory of linear programming. Most prominent among their results was the duality theorem for linear programming: To a linear programming problem one can formulate another linear programming problem on the same set of data called the *dual* program. The *duality theorem* says that the original, or *primal* program has a finite optimal solution if and only if the dual has a finite optimal solution, and the optimum value will be the same [Gale, *et al.*, 1951].

The connection between game theory and linear programming provided the Air Force problem with a mathematical theory, and in so doing changed the scientific status of linear programming. Linear programming was no longer just a practical problem the military wanted solved, but formed part of mathematical disciplines such as linear inequality theory, convex

[7]See [Rees, 1977], [Schweber, 1988] and [Dupree, 1986].

[8]See [Sapolsky, 1979], [Schweber, 1988], [Old, 1961], [Zachary, 1997].

analysis, and game theory. The connection to game theory broadened the subject of linear programming and suggested further mathematical problems [Kjeldsen, 1999b].

This enhanced linear programming's attractiveness as a potential mathematical research area. This change in scientific status was crucial for the further development of mathematical programming, including nonlinear programming.

Until Tucker got involved, the driving forces behind the development were practical applications—the solving of the Air Force programming problem. Tucker, Kuhn, and Gale, on the other hand, worked within a university context, and the duality theorem for linear programming made it an interesting research area in mathematics. And this is where nonlinear programming enters the picture, because what happened was that Kuhn and Tucker never really left the project.

Nonlinear Programming

In the autumn of 1949—that is, a few months after the first conference on linear programming—Tucker went to Stanford on leave. He had time to explore his first intuition about linear programming—the resemblance to Kirchoff-Maxwell's law for electrical networks. He perceived the underlying optimization problem of minimizing heat loss. The objective function was not a linear but a quadratic function. This suggested to Tucker that maybe the Lagrangian multiplier method could be adapted to optimization problems with *inequality constraints* [Kuhn, 1976, p.12–13].

Tucker then wrote to Kuhn and Gale and asked if they were interested in continuing their work to extend the duality theorem for linear programming to quadratic programs [Kuhn, 1976, p.13]. David Gale said no, Kuhn on the other hand said yes. Some way along the working process, Kuhn and Tucker changed the focus from quadratic programs to the general nonlinear case.

The central idea underneath Kuhn and Tucker's development of nonlinear programming was the saddle-point property of the associated Lagrangian function. From the linear programming problem:

$$\text{maximize} \quad g(x) = \sum_{i=1}^{n} c_i x_i, \qquad c_i \in \mathbf{R},$$

subject to

$$f_h(x) = b_h - \sum_{i=1}^{n} a_{hi} x_i \geq 0, \qquad x_i \geq 0,$$
$$h = 1, \ldots, m, \quad i = 1, \ldots, n, \quad a_{hi}, b_h \in \mathbf{R},$$

Kuhn and Tucker formed the corresponding Lagrangian function:

$$\phi(x,u) = g(x) + \sum u_h f_h(x), \qquad x_i \geq 0,\ u_h \in \mathbf{R}.$$

They realized that $x^0 = (x^0{}_1, \ldots, x^0{}_n)$ will maximize $g(x)$ subject to the given constraints if and only if there exists a vector $u^0 \in \mathbf{R}^m$ with non-negative components (multipliers), such that (x^0, u^0) is a saddle-point for the Lagrangian $\phi(x,u)$ [Kuhn and Tucker, 1950, p. 481].

The really neat thing about this saddle-point property was, as Kuhn and Tucker phrased it,

> *The bilinear symmetry of $\phi(x,u)$ in x and u yields the characteristic duality of linear programming.* [Kuhn and Tucker, 1950, p. 481]

If x^0 is a solution to a linear programming problem and (x^0,u^0) is the saddle-point for the corresponding Lagrangian function, then u^0 will be an optimal solution to the dual programming problem. If the object was to extend the duality theory for linear programming to the more general case of nonlinear programs, it would seem natural to take the saddle-point property of the Lagrangian as a starting point. And this was exactly what Kuhn and Tucker did.

They proved that a necessary condition that a point $x^0 \in \mathbf{R}^n$ solve a nonlinear programming problem is the existence of a point (multipliers) $u^0 \in \mathbf{R}^m$, with the property that (x^0,u^0) satisfy the necessary conditions for being a saddle point for the corresponding Lagrangian function. This result came to be the celebrated Kuhn-Tucker theorem. By requiring concavity and differentiability of the functions involved, the objective as well as the constraint functions, Kuhn and Tucker obtained complete equivalence between solutions to a nonlinear programming problem and saddle points for the corresponding Lagrangian [Kuhn and Tucker, 1950].

The conditions for being a saddle-point for the Lagrangian, that is, the necessary conditions for being a solution to a nonlinear programming problem, are now called the Kuhn-Tucker conditions. The theorem about necessary conditions is called the Kuhn-Tucker theorem—or more properly the Karush-Kuhn-Tucker theorem.[9] That theorem launched the mathematical theory of nonlinear programming.

Conclusion

It was the duality theorem for linear programming—that is, a purely theoretical result—that sparked the interest of Kuhn and Tucker. It was the duality theory they wanted to extend to the general (quadratic) nonlinear case. It is in this respect that I find the development of the duality theorem in linear programming so crucial for the emergence of nonlinear programming.

Even though nonlinear programming originated in a context of linear programming, the driving force behind Kuhn and Tucker's development of nonlinear programming was indeed very different from the stimulus that started the development of linear programming. Linear programming originated in the context of the concrete solving of a practical problem within the Air Force. Nonlinear programming on the other hand developed in accordance with the inner rules for research in mathematics proper as it is typically done in a university setting. The appearance of a concrete logistic problem played a decisive role in the origin of linear programming, but Kuhn and Tucker's research into nonlinear programming was not motivated by a problem of this kind. The ONR did not provide Kuhn and Tucker with a concrete practical problem that they required to be solved. What they did provide was financial support for mathematical research—applied as well as pure—as

[9] William Karush proved a version of the theorem that later got known as the Kuhn-Tucker theorem in his master's thesis from 1939 [Karush, 1939]. For a contextualized historical analysis of the aspect of the multiple discovery in connection with the Kuhn-Tucker theorem see [Kjeldsen, 1999c].

long as it was related to optimization. This covered research in areas like game theory, linear inequality theory, theory of convexity, and mathematical programming. It was in that spirit that Kuhn and Tucker's work on nonlinear programming emerged.

Even though Kuhn in his daily work did not feel the presence of the military, one must say that the Office of Naval Research had an enormous influence on the origin of nonlinear programming. From the perspective of the ONR the potential applicability was decisive for the origin of the project. Tucker's project was not a product of university-based research in mathematics proper, but came into being on the initiative of ONR, who ordered the research in these areas. ONR's influence on mathematical research areas like game theory and mathematical programming remained strong. They continued to support Tucker's project until 1972, when the National Science Foundation took over.

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