Why Parrondo’s Paradox Is Irrelevant for Utility Theory, Stock Buying, and the Emergence of Life

Can Losses Be Combined to Give Gains?

INTRODUCTION

Since the initial publication of Parrondo’s paradox [1, 2], a number of articles on the subject have appeared both in the academic literature and in the media. The paradox describes a reversal of fortune for a gambler who loses money by playing either of two games of chance, but who makes money by randomly playing the two games. The academic literature suggests that Parrondo’s paradox advances von Neumann and Morgenstern’s work on utility theory, that it offers money for free, that it has implications for investing strategies, and that it may hold the key for understanding certain biological processes, possibly even the emergence of life [3, 4]. Harmer and Abbott [2] suggest that the paradox might operate in economics or social dynamics to extract benefits from ostensibly detrimental situations. For example, if a society or an ecosystem suffers from declines in either the birth rate or the death rate, declines in both together might combine with favorable consequences.

Media reports compare Parrondo’s paradox with the sacrifice of chess pieces to win a game. There are suggestions that Parrondo’s paradox offers the possibility of making money by investing in losing stocks. Paulos [5] observes that “standard stock market investments cannot be modeled by games of this type, but variations of these games might conceivably give rise to counterintuitive investment strategies. Although a much more complex phenomenon, the ever-increasing valuations of some dot-coms with continuous losses may not be as absurd as they seem. Perhaps they’ll one day be referred to as Parrondo profits.” The same article quotes Derek Abbott as saying that “President Clinton, who at first denied having a sexual affair with Monica S. Lewinsky saw his popularity rise when he admitted that he had lied. The added scandal created more good for Mr. Clinton.” All these examples seem to suggest miraculous possibilities for turning unfavorable situations into favorable ones, either by purposeful actors or by acts of nature. Parrondo’s paradox appears to be a remarkable finding with far-reaching consequences in wide ranging situations.

Or is it? That’s the question we ask and hope to answer. To do so, we need to look more closely at the rationale behind Parrondo’s paradox, stripping it of all extraneous baggage and descriptive analogies as ratchets and pawls. We need to understand...
the structures of the associated games of chance and uncover what exactly drives the reversal of fortune in Parrondo’s games. We will then be in a better position to assess if one or another phenomenon can be explained by it and what opportunities it offers to investors and gamblers, biologists and economists.

THE LOGIC OF PARRONDO’S PARADOX

Parrondo’s paradox is typically described using two games. One game is a simple coin toss—a zero-order (memoryless) game. The other game has the structure of a Markov chain—it uses several coins, one of which is chosen for play depending on what has happened on previous coin tosses. The expected return from the zero-order game depends only on the probability of landing heads. In contrast, the expected return from the history-dependent game depends not only on the probabilities with which the various coins land heads but also on how often each coin is tossed, which is determined by the steady-state probabilities of the Markov chain.

The games are set up so that if played separately, each game has a negative expected return. Consider randomly switching between the games. What happens? The answer is that randomization changes the transition probabilities and, therefore, the steady-state probabilities for the Markov-chain game. This in a nutshell is the reasoning behind the paradox: an unconditional probability (the probability of winning the memoryless game) enters as a conditional (transition) probability in the Markov-chain game, changing the steady-state probabilities of the latter and reversing a player’s fortune.

Another interpretation of the paradox is as follows. Consider playing only the history-dependent game; it leads to losses. Now add an option to each state of the history-dependent game: allow the player the option of tossing another coin. Surely the option is worth using only if the new coin gives better odds in at least some states of the history-dependent game. Now suppose we impose the constraint that a player must use the new coin with the same probability in all states of the history-dependent game. Then it is possible that, even if the new coin offers unfavorable odds, the modified game has winning odds. The demonstration of this feasibility is Parrondo’s paradox.

A reasonable question to ask is if there is any need to randomize at all. Before each play, why not just compare the odds offered by the zero-order game with the odds offered by the Markov chain game? You could simply play the game that offered better odds. The answer is that this is indeed a dominant strategy. It is always better to do this than to randomize over games.

We formalize this argument below. For illustrative purposes, we consider the more recent capital-independent games [6], which use a second-order Markov chain to describe the state-dependent game. The same line of argument can be adapted in an obvious way to games in which the payoffs from one game depend on the capital. We then describe a very simple game based on a first-order Markov chain for which a strategic play, but not randomization, produces winning odds. We conclude by reassessing the relevance of Parrondo’s paradox for economic theory, for strategies of investments in stocks, and for economics or social dynamics that “extract benefits from ostensibly detrimental situations” [1].

CAPITAL INDEPENDENT GAMES

Consider two coin-toss games, A and B. All bets are a dollar per play. A player loses a dollar if a tossed coin lands tails and wins a dollar if it lands heads. Game A is played by tossing coin $A_i$ that lands heads with probability $p_i < 1/2$; the long-run returns from game A are negative. Game B, which also has negative long-run returns, is played with 4 coins; coin $B_i$ lands heads with probabilities $p_i$, $1 \leq i \leq 4$. Game B is played as follows:

<table>
<thead>
<tr>
<th>If the two preceding outcomes are</th>
<th>then toss coin</th>
</tr>
</thead>
<tbody>
<tr>
<td>Both losses (LL)</td>
<td>$B_1$</td>
</tr>
<tr>
<td>A loss followed by a win (LW)</td>
<td>$B_2$</td>
</tr>
<tr>
<td>A win followed by a loss (WL)</td>
<td>$B_3$</td>
</tr>
<tr>
<td>Both wins (WW)</td>
<td>$B_4$</td>
</tr>
</tbody>
</table>

Note that Game B has the structure of the following second-order Markov Chain:

$$A = \begin{pmatrix} LL & LW & WL & WW \\ q_1 & p_1 & 0 & 0 \\ 0 & 0 & q_2 & p_2 \\ 0 & 0 & q_3 & p_3 \end{pmatrix}$$

where $q_i = 1 - p_i$, $1 \leq i \leq 4$. The row represents the four possible outcomes, LL, LW, WL, WW, on the two preceding plays, say plays $(j − 1)$ and $j$; the column also represents the same four outcomes on plays $j$ and $(j + 1)$. Thus, play $j$ is common to the row and column states. The cell entries are the probabilities of transition from a row state to a column state. The nonzero entries are the probabilities associated with the outcomes on plays $(j − 1)$ and $j$.

Let $\pi_1$, $\pi_2$, $\pi_3$, $\pi_4$ denote the steady-state probabilities associated with the states LL, LW, WL, WW. Then [6]:

$$\pi_1 = \frac{q_1 q_4}{N},$$

$$\pi_2 = \pi_3 = \frac{p_1 q_4}{N},$$

$$\pi_4 = \frac{p_1 p_2}{N},$$

$$N = q_1 q_4 + p_1 p_2 + 2 p_1 q_4.$$ 

Thus, for example, $\pi_1$ is the long-run probability of two successive losses, LL. It follows that the unconditional probability of winning (i.e., landing heads) on any play is

$$p_{\text{win}} = \pi_1 p_1 + \pi_2 p_2 + \pi_3 p_3 + \pi_4 p_4,$$

which can be written as

$$p_{\text{win}} = \frac{1}{2 + (c/s)} c = q_1 q_4 - p_1 p_2,$$

$$s = p_1 (p_2 + q_4).$$
Thus, game $B$ gives losing returns in the long run if $p_{\text{win}} < 1/2$ or equivalently if $c > 0$. To change a losing game $B$ into a winning game, one requires $c < 0$. Suppose we play $A$ and $B$ at random, selecting game $A$ with probability $\lambda$ and game $B$ with probability $(1 - \lambda)$. Game $B$ requires tossing:

1. coin $B_1$ if the two preceding outcomes are both losses (LL).
2. coin $B_2$ if the last two outcomes were a loss followed by a win (LW);
3. coin $B_3$ after a win followed by a loss (WL); and
4. coin $B_4$ after a pair of wins (WW).

We can write the corresponding Markov chain by replacing $p_i$ by $p' = \lambda p + (1 - \lambda)p_i$, $1 \leq i \leq 4$, in $A$. The unconditional probability of winning (landing heads) is therefore

$$p'_{\text{win}} = \frac{1}{2 + (c/s)} c' = q_1q_4 - p_1p_2.$$  

The revised odds of winning exceed one half if $c' < 0$, which is possible even if $c > 0$. This is the reversal of fortune in Parrondo’s paradox.

Is this the best way for a gambler to combine the two games? The answer is “no.” Randomizing over the two games forces the same convex combination of $p$ and $p_i$, $1 \leq i \leq 4$. This is a natural constraint for Brownian ratchets, the original motivation for the paradox, because one cannot separate particles by their states; the only way to affect the transition probabilities is to randomly change the potential applied to the entire system of particles. But there is no such natural constraint for a gambler who will switch from one game to the other whenever it is profitable to do so. This trivial solution is the best for precisely the same reason that randomization works, except that it forces a player to choose a fixed probability for playing the two games regardless of the state of game $B$. In both cases, the change in fortune is brought about by a change in the transition probabilities of a Markov chain. Suppose we allow different values $\lambda_i$ for each $i$ in the expression $\lambda_i p + (1 - \lambda_i)p_i$, $1 \leq i \leq 4$; as $c'$ decreases with $p_1$, $p_2$, $p_3$, $p_4$, its lower bound is obtained by setting $\lambda_i = 1$ when $p_i < p$ and $\lambda_i = 0$ when $p_i > p$ (for $p_i = p$, $\lambda_i$ can take any value between zero and one). The corresponding deterministic strategy gives a winning probability $p'^*_{\text{win}} = p'_{\text{win}}$ where $p' = \max(p, p_i)$ and

$$p'^*_{\text{win}} = \frac{1}{2 + (c/s)} c^* = q_1q_4 - p_1p_2.$$  

As $\lambda p + (1 - \lambda)p_i \equiv \max(p, p_i)$, we have $c^* \leq c'$, the equality occurring only when $p_i = p$, $1 \leq i \leq 4$. Otherwise, the lower bound $c^*$ is achieved by playing game $A$ if $p_i < p$, and playing game $B$ otherwise. This obvious and smart way of combining the two games produces winning odds whenever randomization does and also in instances when randomization does not yield a winning game.

To summarize, the best possible way of combining games $A$ and $B$ in Parrondo et al. [6] is to play game $A$ whenever it offers higher winning odds than does the relevant coin in Game $B$; otherwise play Game $B$. Unless game $A$ always dominates game $B$, the winning odds from this strategy always increase if $p_i < p$ for some $1 \leq i \leq 4$; and they exceed one half if $c < 0$, which is equivalent to the condition $p_1p^*_2 > q_1q^*_4$. This trivial “optimal” strategy suggests that randomization should not be used to play the games of Parrondo’s paradox unless a player is uninformed about the states (the sequence of outcomes of plays) and therefore does not know when to switch to the single-coin game. But it is not clear to us when and if such games arise in games of chance.

It is also important to note that game $B$ above has the structure of a second-order Markov chain. It is for this specific structure, and for higher-order Markov chains, that one attains Parrondo’s paradox. But randomization cannot produce winning combinations of two losing games if game $B$ is described by a first-order Markov chain or if it is also a zero-order game (i.e., just a simple coin toss). We will examine these cases below. But it is worth noting that any situation in which Parrondo paradox appears should have the appropriate structure—the transitions between outcomes should be representable in the form of at least a second-order Markov chain. It is difficult to see how public attitude towards a past President, stock investments and the many other cases in which Parrondo’s paradox is evoked [3, 4] can be represented as a second-order Markov chain. One cannot take just any two uncertain, losing opportunities and hope that randomizing over them will produce winning odds. We show this below, where we consider combining a simple coin toss with a game represented by a first-order Markov chain. We show that a randomization strategy over two such games cannot produce a winning combination. In contrast, the simpler deterministic switching noted above can produce a winning combination in this situation.

### First-Order Markov Games

Consider two games $A$ and $B$. All bets are a dollar per play. A player loses a dollar if a tossed coin lands tails and wins a dollar if it lands heads. Game $A$ is unchanged—it is played with a single coin and is winning with probability $p < 1/2$. Game $B$ is played with two coins, $B_1$ and $B_2$. Coin $B_1$ is tossed after a win; it leads to a win (loss) with probability $p_1(q_1 = 1 - p_1)$. Coin $B_2$ is tossed after a loss; it leads to a win (loss) with probability $p_2(q_2 = 1 - p_2)$. These probabilities can be represented by the following first-order Markov chain:

$$W \quad L$$  

$$\begin{pmatrix} W & p_1 & q_1 \\ L & p_2 & q_2 \end{pmatrix}.$$  

The corresponding steady-state probability of a win is $p_{\text{win}} = p_2/(q_1 + p_2)$.
which is increasing in $p_1$ and $p_2$. Game $B$ is losing if $p_{\text{win}} < 1/2$, or equivalently if $p_2 < q_1$. Let $p > p_2$. A randomization strategy in which game $A$ is played with probability $\lambda$ and game $B$ is played with probability $1 - \lambda$ leads to the transition matrix,

$$
\begin{bmatrix}
W & L \\
L & \begin{pmatrix} p_1' & q_1' \\
p_2' & q_2' \end{pmatrix}
\end{bmatrix}
$$

where $p_1' = \lambda p + (1 - \lambda) p_1$, $i = 1, 2$. The probability of winning is $p_{\text{win}} = p_2'$, $i = 1, 2$. Thus, $p_{\text{win}} > 1/2$ implies $p_2' > q_1'$. Substituting for $p_1'$ gives the condition $\lambda(p_1 + p_2) > 1$.

The steady-state probability of winning, $p$, exceeds one half if $p > p_1$, which is feasible. For example, $p = 0.45$, $p_1 = 0.6$, $p_2 = 0.3$ gives $p_{\text{win}} = 3/7$ for game $B$, so both $A$ and $B$ are losing games. However, playing game $A$ whenever game $B$ allows tossing only $B_1$ gives the winning odds of 9/17.

Finally, consider two games that are both described by first-order Markov chains. Once again, all bets are a dollar per play; a player loses a dollar if a-tossed coin lands tails and wins a dollar if it lands heads. In game $A$, the probability of winning (losing) after a win is $p_1(q_1)$ and of winning (losing) after a loss is $p_2(q_2)$. In game $B$, the probability of winning (losing) after a win is $p_3(q_3)$ and of winning (losing) after a loss is $p_4(q_4)$. Game $A$ is losing if $p_2 < q_1$, and game $B$ is losing if $p_1 < q_2$. Suppose $p_1 < p_3$ and $p_4 < p_2$. The transition matrix if game $A$ is played with probability $\lambda$ and game $B$ is played with probability $1 - \lambda$ is

$$
\begin{bmatrix}
W & L \\
L & \begin{pmatrix} \lambda p_1 + (1 - \lambda)p_3 & 1 - (\lambda p_1 + (1 - \lambda)p_3) \\
\lambda p_2 + (1 - \lambda)p_4 & 1 - (\lambda p_2 + (1 - \lambda)p_4) \end{pmatrix}
\end{bmatrix}
$$

Probabilistic play (randomization) is a winning strategy if $\lambda(p_1 + p_2) > 1$, which is impossible because $p_2 < q_1$ and $p_4 < q_2$. Now suppose game $A$ is played after a loss and game $B$ is played after a win. Then the corresponding transition matrix is

$$
\begin{bmatrix}
W & L \\
L & \begin{pmatrix} p_1 & q_1 \\
p_2 & q_2 \end{pmatrix}
\end{bmatrix}
$$

The steady-state probability of winning is $p_2/(q_3 + p_2)$, which exceeds one half if $q_3 < p_2$. Thus, playing game $A$ after a loss and game $B$ after a win gives better than all odds of winning if $p_4 < q_1 < p_2 < q_1$. For example, $p_1 = 0.2$, $p_2 = 0.75$, $p_3 = 0.4$, $p_4 = 0.5$, gives the winning odds 15/31 for game $A$, 5/11 for game $B$, and 15/27 for strategic play.

**CONCLUSION**

There are deterministic ways of playing the games of Parrondo’s paradox that never do worse than randomization and can provide better winning odds under more general conditions. This happens because randomization forces a constraint on the modified transition probability matrices of the history-dependent game. Removing the constraint improves winning odds and leads to a strategy in which the single-coin game is played whenever it offers better odds than the coin one is forced to toss in the history-dependent game. One should note that there is a difference between Brownian ratchets and gambles of the sort considered here and by Harmer and Abbott [1, 2] and Parrondo et al. [6]. In a Brownian ratchet particles can be subjected to either a flat potential or an asymmetric potential. One cannot separate particles by their states. Thus, one cannot deterministically change the transition probabilities for some particles but not for others. Such a constraint does not naturally occur in games of chance. A player offered the choice of two games can switch from one to the other whenever it is profitable to do so. This strategy dominates randomization.

It is incorrect to suggest that Parrondo’s paradox represents an advance in utility theory [6]. Utility theory specifies a set of preference axioms for which a decision maker maximizes expected utility over risky outcomes. Neither randomization nor a deterministic betting strategy is an extension of utility theory because strategies have associated payoffs with which a player attaches utility. So long as utility is monotonically increasing in payoffs, a utility-maximizing player will prefer higher payoffs to lower ones. The fact that randomization gives positive returns means that a risk-neutral player will prefer to play a randomized game than to not play it (and prefer to play the trivial deterministic strategy over a randomized game), and this is perfectly consistent with utility theory, not an extension or departure from it. Parrondo’s games have also been incorrectly said to be paradoxes of game theory [1, 2], despite the fact that these are games against nature; game theory, in contrast, is concerned with the actions and reactions of strategic players. Games of chance, of which Parrondo’s games are examples, are quite different because there is no active ad-
versary whose behavior changes with the opponents’ play.

One view that has caught the imagination of many people is that Parrondo’s paradox suggests the possibility of making money by investing in losing stocks. The above analysis makes clear why such a conclusion is unwarranted. Even if stock markets were exactly modeled by the types of games considered by Parrondo, the strategy to use would be the obvious one suggested by the above analysis: sell stock that has particularly poor prospects and buy stock with better prospects; this way you will always do better than if you randomize over losing stocks.

Parrondo’s paradox is suggested to possibly operate in economics or social dynamics to extract benefits from ostensibly detrimental situations. Harmer and Abbott [1] suggest the example of declining birth and death rates. Each by itself has a “negative” effect on a society or an ecosystem, but declines in both together might combine with “favorable consequences.” It is unclear what Harmer and Abbott have in mind. First, it is not clear what negative and favorable consequences mean; and second, there is no reason, as far as we know, to believe that one or both of the rates of births and deaths have the structure of a Markov chain or that nature can somehow randomize between births and deaths, to create a realization of Parrondo’s paradox. In a similar vein, Parrondo’s paradox has been compared with the sacrifice of chess pieces to win a game, to the decline in the fitness of a species before it evolves a higher level of survival fitness, to the stabilization of unstable systems when combined in the right way, and to public attitudes toward politicians. As best as we know, none of these is related to the fact that changing the transition probabilities of a Markov chain affects the steady-state probabilities associated with the states, which in the end is the entire reason for Parrondo’s paradox.

REFERENCES