$f^* = m/(2(2c+m))$. We see that for m fixed, as c decreases from $(1-m)/2$ and $\text{cor}(X_1, X_2) = 1$, to 0 and $\text{cor}(X_1, X_2) = -(1-m)/(1+m)$, f^* for each bet increases from $m/2$ to $1/2$, as in Table 6.2.

TABLE 6.2 f^* increases as Cor (X_1, X_2) decreases.

$\text{Cor}(X_1, X_2)$	c	f^*
1	$(1-m)/2$	$m/2$
0	$(1-m^2)/4$	$m/(1+m^2)$
$-(1-m)/(1+m)$	0	$1/2$

It is important to note that for an exact solution or an arbitrarily accurate numerical approximation to the simultaneous bet problem, covariance or correlation information is not enough. We need to use the entire joint distribution to construct the g function.

We stopped sports betting after our successful test for reasons including: (1) It required a person on site in Nevada. (2) Large amounts of cash and winning tickets had to be transported between casinos. We believed this was very risky. To the sorrow of others, subsequent events confirmed this. (3) It was not economically competitive with our other operations.

If it becomes possible to place bets telephonically from out of state and to transfer the corresponding funds electronically, we may be back.

7 Wall Street: the biggest game

To illustrate both the Kelly criterion and the size of the securities markets, we return to the study of the effects of correlation as in Example 6.2. Consider the more symmetric and esthetically pleasing pair of bets U_1 and U_2, with joint distribution given in Table 7.1

TABLE 7.1 Joint distribution of U_1 and U_2.

U_1 :	U_2 : m_2+1	m_2-1
m_1+1	a	$1/2-a$
m_1-1	$1/2-a$	a

Clearly $0 \leq a \leq 1/2$ and $\mathrm{Cor}(U_1, U_2) = \mathrm{Cor}(U_1, U_2) = 4a - 1$ increases from -1 to 1 as a increases from 0 to 1/2. Finding a general solution for (f_1^*, f_2^*) appears algebraically complicated (but specific solutions are easy to find numerically), which is why we chose Example 6.2 instead. Even with reduction to the special case $m_1 = m_2 = m$ and the use of symmetry to reduce the problem to finding $f^* = f_1^* = f_2^*$, a general solution is still much less simple. But consider the instance when $a = 0$ so $\mathrm{Cor}(U_1, U_2) = -1$. then $g(f) = \ln(1 + 2mf)$ which increases without limit as f increases. This pair of bets is a "sure thing" and one should bet as much as possible.

This is a simplified version of the classic arbitrage of securities markets: find a pair of securities which are identical or "equivalent" and trade at disparate prices. Buy the relatively underpriced security and sell short the relatively overpriced security, achieving a correlation of -1 and "locking in" a riskless profit. An example occured in 1983. My investment partnership bought \$ 330 million worth of "old" AT & T and sold short \$ 332.5 million worth of when-issued "new" AT & T plus the new "seven sisters" regional telephone companies. Much of this was done in a single trade as part of what was then the largest dollar value block trade ever done on the New York Stock Exchange (December 1, 1983).

In applying the Kelly criterion to the securities markets, we meet new analytic problems. A bet on a security typically has many outcomes rather than just a few, as in most gambling situations. This leads to the use of continuous instead of discrete probability distributions. We are led to find f to maximize $g(f) = Eln(1 + fX) = \int ln(1 + fx)dP(x)$ where $P(x)$ is a probability measure describing the outcomes. Frequently the problem is to find an optimum portfolio from among n securities, where n may be a "large" number. In this case x and f are n-dimension vectors and fx is their scalar product. We also have constraints. We always need $1 + fx > 0$ so $ln(\cdot)$ is defined, and $\sum f_i = 1$ (or some $c > 0$) to normalize to a unit (or to a $c > 0$) investment. The maximization problem is generally solvable because $g(f)$ is concave. There may be other constraints as well for some or all i such as $f_i \geq 0$ (no short selling), or $f_i \leq M_i$ or $f_i \geq m_i$ (limits amount invested in ith security), or $\sum \mid f_i \mid \leq M$ (limits total leverage to meet margin regulations or capital requirements). Note that in some instances there is not enough of a good bet or investment to allow betting the full f^*, so one is forced to underbet, reducing somewhat both the overall growth rate and the risk. This is more a problem in the gaming world than in the much larger securities markets. More on these problems and techniques may be found in the literature.

(a) **Continuous approximation.**

There is one technique which leads rapidly to interesting results. Let X be a random variable with $P(X = m + s) = P(X = m - s) = 0.5$. Then $E(X) = m$, $\mathrm{Var}(X) = s^2$. With initial capital V_0, betting fraction f, and return per unit of X, the result is

$$V(f) = V_0(1 + (1 - f)r + fX) = V_0(1 + r + f(X - r)),$$

where r is the rate of return on the remaining capital, invested in, e.g., Treasury bills. Then

$$\begin{aligned} g(f) &= E(G(f)) = E(\ln(V(f)/V_0)) = E\ln(1 + r + f(X - r)) \\ &= 0.5\ln(1 + r + f(m - r + s)) + 0.5\ln(1 + r + f(m - r - s)). \end{aligned}$$

Now subdivide the time interval into n equal independent steps, keeping the same drift and the same total variance. Thus m, s^2 and r are replaced by m/n, s^2/n and r/n, respectively. We have n independent X_i, $i = 1, \ldots, n$, with

$$P\left(X_i = m/n + sn^{-1/2}\right) = P\left(X_i = m/n - sn^{-1/2}\right) = 0.5$$

Then

$$V_n(f)/V_0 = \prod_{i=1}^{n}(1 + (1 - f)r + fX_i)$$

Taking $E(\log(\cdot))$ of both sides gives $g(f)$. Expanding the result in a power series leads to

$$g(f) = r + f(m - r) - s^2f^2/2 + 0\left(n^{-1/2}\right) \tag{7.1}$$

where $0(n^{-1/2})$ has the property $n^{1/2}0(n^{-1/2})$ is bounded as $n \to \infty$. Letting $n \to \infty$ in (7.1) we have

$$g_\infty(f) \equiv r + f(m - r) - s^2f^2/2 \tag{7.2}$$

The limit $V \equiv V_\infty(f)$ of $V_n(f)$ as $n \to \infty$ corresponds to a log normal diffusion process, which is a well-known model for securities prices. The "security" here has instantaneous drift rate m, variance rate s^2, and the riskless investment of "cash" earns at an instanteous rate r. Then $g_\infty(f)$ in (7.2) is the (instantaneous) growth rate of capital with investment or betting fraction

f. There is nothing special about our choice of the random variable X. Any bounded random variable with mean $E(X) = m$ and variance $\operatorname{Var}(X) = s^2$ will lead to the same result. Note that f no longer needs to be less than or equal to 1. The usual problems, with $\log(\cdot)$ being undefined for negative arguments, have disappeared. Also, $f < 0$ causes no problems. This simply corresponds to selling the security short. If $m < r$ this could be advantageous. Note further that the investor who follows the policy f must now adjust his investment "instantaneously". In practice this means adjusting in tiny increments whenever there is a small change in V. This idealization appears in option theory. It is well known and does not prevent the practical application of the theory (Black and Scholes, 1973). Our previous growth functions for finite sized betting steps were approximately parabolic in a neighborhood of f^* and often in a range up to $0 \le f \le 2f^*$, where also often $2f^* \doteq f_c$. Now with the limiting case (7.2), $g_\infty(f)$ is exactly parabolic and very easy to study.

Lognormality of $V(f)/V_0$ means $\log(V(f)/V_0)$ is $N(M, S^2)$ distributed, with mean $M = g_\infty(f)t$ and variance $S^2 = \operatorname{Var}(G_\infty(f))t$ for any time t. From this we can determine, for instance, the expected capital growth and the time t_k required for $V(f)$ to be at least k standard deviations above V_0. First, we can show by our previous methods that $\operatorname{Var}(G_\infty(f)) = s^2f^2$, hence $\operatorname{Sdev}(G_\infty(f)) = sf$. Solving $t_k g_\infty = k t_k^{1/2} \operatorname{Sdev}(G_\infty(f))$ gives $t_k g_\infty^2$ hence the expected capital growth $t_k g_\infty$, from which we find t_k. The results are summarized in equations (7.3).

$$f^* = (m-r)/s^2 \qquad g_\infty(f) = r + f(m-r) - s^2f^2/2 \tag{7.3}$$

$$\begin{aligned}
g_\infty(f^*) &= (m-r)^2/2s^2 + r \\
\operatorname{Var}(G_\infty(f)) &= s^2f^2 \qquad \operatorname{Sdev}(G_\infty(f)) = sf \\
t_k g_\infty(f) &= k^2s^2f^2/g_\infty \\
t_k &= k^2s^2f^2/g_\infty^2
\end{aligned}$$

Examination of the expressions for $t_k g_\infty(f)$ and t_k show that each one increases as f increases, for $0 \le f < f_+$ where f_+ is the positive root of $s^2f^2/2 - (m-r)f - r = 0$ and $f_+ > 2f^*$.

Comment: The capital asset pricing model (CAPM) says that the market portfolio lies on the Markowitz efficient frontier E in the (s, m) plane at a (generally) unique point $P = (s_0, m_0)$ such that the line determined by P

and $(s = 0, m = r)$ is tangent to E (at P). The slope of this line is the Sharpe ratio $S = (m_0 - r_0)/s_0$ and from (7.3) $g_\infty(f^*) = S^2/2 + r$ so the maximum growth rate $g_\infty(f^*)$ depends, for fixed r, only on the Sharpe ratio. (See Quaife (1995). Again from (7.3), $f^* = 1$ when $m = r + s^2$ in which case the Kelly investor will select the market portfolio without borrowing or lending. If $m > r + s^2$ the Kelly investor will use leverage and if $m < r + s^2$ he will invest partly in T-bills and partly in the market portfolio. Thus the Kelly investor will dynamically reallocate as f^* changes over time because of flunctuations in the forecast m, r and s^2, as well as in the prices of the portfolio securities.

From (7.3), $g_\infty(1) = m - s^2/2$ so the portfolios in the (s, m) plane satisfying $m - s^2/2 = C$, where C is a constant, all have the same growth rate. In the continuous approximation, the Kelly investor appears to have the utility function $U(s, m) = m - s^2/2$. Thus, for any (closed, bounded) set of portfolios, the best portfolios are exactly those in the subset that maximizes the one parameter family $m - s^2/2 = C$. See Kritzman in Bernstein and Damodaran editors (1998), Chapter 2, for an elementary introduction to related ideas.

Example 7.1. The long run revisited. For this example let $r = 0$. Then the basic equations (7.3) simplify to

$$(7.4) \qquad \begin{gathered} r = 0: \quad f^* = m/s^2 \quad g_\infty(f) = mf - s^2f^2/2 \\ g_\infty(f^*) = m^2/2s^2 \\ \operatorname{Var}(G_\infty(f)) = s^2f^2 \qquad \operatorname{Sdev}(G_\infty(f)) = sf \end{gathered}$$

How long will it take for $V(f^*) \geq V_o$ with a specified probability? How about $V(f^*/2)$? To find the time t needed for $V(f) \geq V_0$ at the k standard deviations level of significance ($k = 1$, $P = 84\%$; $k = 2$, $P = 98\%$, etc.) we solve for $t \equiv t_k$:

$$(7.5) \qquad tg_\infty(f) = kt^{1/2}\operatorname{Sdev}(G_\infty(f))$$

We get more insight by normalizing all f with f^*. Setting $f = cf^*$ throughout, we find when $r = 0$

$$(7.6) \qquad \begin{gathered} r = 0: \qquad f^* = m/s^2 \qquad f = cm/s^2 \\ g_\infty(cf^*) = m^2(c - c^2/2)/s^2 \end{gathered}$$

$$\begin{aligned} \operatorname{Sdev}(G_\infty(cf^*)) &= cm/s \\ tg_\infty(cf^*) &= k^2c/(1 - c/2) \\ t(k, cf^*) &= k^2s^2/\left(m^2(1 - c/2)^2\right) \end{aligned}$$

Equations (7.6) contain a remarkable result: $V(f) \geq V_0$ at the k standard deviation level of significance occurs when expected capital growth $tg_\infty = k^2c/(1-c/2)$ and this result is *independent of* m *and* s. For $f = f^*$ ($c = 1$ in (7.6)), this happens for $k = 1$ at $tg_\infty = 2$ corresponding to $V = V_0e^2$ and at $k = 2$ for $tg_\infty = 8$ corresponding to $V = V_0e^8$. Now $e^8 \doteq 2981$ and at a 10% annual (instantaneous) growth rate, it takes 80 years to have a probability of 98 % for $V \geq V_0$. At a 20% annual instantaneous rate it takes 40 years. However, for $f = f^*/2$, the number for $k = 1$ and 2 are $tg_\infty = 2/3$ and $8/3$, respectively, just 1/3 as large. So the waiting times for Prob$(V \geq V_0)$ to exceed 84% and 98% become 6.7 years and 26.7 years, respectively, and the expected growth rate is reduced to 3/4 of that for f^*.

Comment: Fractional Kelly versus Kelly when $r = 0$.

From equations (7.6) we see that $g_\infty(cf^*)/g_\infty(f^*) = c(2-c)$, $0 \leq c < \infty$, showing how the growth rate relative to the maximum varies with c. The relative risk Sdev$(G_\infty(cf^*))$/Sdev$(G_\infty(f^*)) = c$ and the relative time to achieve the same expected total growth is $1/c(2-c)$, $0 < c < 2$. Thus the relative "spread" for the same expected total growth is $1/(2-c)$, $0 < c < 2$. Thus, even by choosing c very small, the spread around a given expected growth cannot be reduced by 1/2. The corresponding results are not quite as simple when $r > 0$.

(b) **The (almost) real world.**

Assume that prices change "continuously" (no "jumps"), that portfolios may be revised " continuously", and that there are no transactions costs (market impact, commissions, "overhead"), or taxes (Federal, State, city, exchange, etc.). Then our previous model applies.

Example 7.2. The S & P 500 Index. Using historical data we make the rough estimates $m = .11$, $s = .15$, $r = .06$. The equations we need for $r \neq 0$ are the generalizations of (7.6) to $r \neq 0$ and $f = cf^*$, which follow from (7.3):

$$cf^* = c\,(m-r)\,/s^2 \tag{7.7}$$

$$\begin{aligned}
g_\infty\,(cf^*) &= \left((m-r)^2\left(c - c^2/2\right)\right)/s^2 + r \\
\text{Sdev}\,(G_\infty\,(cf^*)) &= c\,(m-r)\,/s \\
tg_\infty\,(cf^*) &= k^2c^2/\left(c - c^2/2 + rs^2/(m-r)^2\right) \\
t\,(k,\, cf^*) &= k^2c^2\left((m-r)^2/s^2\right)/\left(\left((m-r)^2/s^2\right)\left(c - c^2/2\right) + r\right)^2
\end{aligned}$$

If we define $\tilde{m} = m - r$, $\tilde{G}_\infty = G_\infty - r$, $\tilde{g}_\infty = g_\infty - r$, then substitution into equations (7.7) give equations (7.6), showing the relation between the two sets. It also shows that examples and conclusions about $P(V_n > V_0)$ in the $r = 0$ case are equivalent to those about $P(\ln(V(t)/V_0) > rt)$ in the $r \neq 0$ case. Thus we can compare various strategies versus an investment compounding at a constant riskless rate r such as zero coupon U.S. Treasury bonds.

From equations (7.7) and $c = 1$, we find

$$f^* = 2.2\bar{2}, \qquad g_\infty(f^*) = .11\bar{5}, \qquad \mathrm{Sdev}(G_\infty(f^*)) = .3\bar{3}$$

$$tg_\infty(f^*) = 0.96k^2 \qquad t = 8.32k^2 \text{ years}$$

Thus, with $f^* = 2.2\bar{2}$, after 8.32 years the probability is 84% that $V_n > V_0$ and the expected value of $\log(V_n/V_0) = .96$ so the median value of V_n/V_0 will be about $e^{.96} = 2.61$.

With the usual unlevered $f = 1$, and $c = 0.45$, we find

$$g_\infty(1) = m - s^2/2 = .09875 \qquad \mathrm{Sdev}\,(G_\infty(1)) = 0.15$$

$$tg_\infty(1) = 0.22k^2 \qquad t(k,\, .45f^*) = 2.31k^2 \quad \text{years.}$$

Writing $tg_\infty = h(c)$ as

$$h(c) = k^2/\left(1/c + rs^2/\left((m-r)^2 c^2\right) - 1/2\right)$$

we see that $h(c)$ increases as c increases, at least up to the point $c = 2$, corresponding to $2f^*$.

Writing $t(k,\, cf^*) = t(c)$ as

$$t(c) = k^2\left((m-r)^2/s^2\right)/\left((m-r)^2/s^2\right)(1-c/2) + r/c^2$$

shows that $t(c)$ also increases as c increases, at least up to the point $c = 2$. Thus for smaller (more conservative) $f = cf^*$, $c \leq 2$, specified levels of $P(V_n > V_0)$ are reached earlier. For $c < 1$, this comes with a reduction in growth rate, which reduction is relatively small for f near f^*.

Note: During the period 1975-1997 the short term T-bill total return for the year, a proxy for r if the investor lends (i.e. $f < 1$), varied from a low of 2.90% (1993) to a high of 14.71% (1981). For details, see Ibbotson Associates 1998 (or latest available) Yearbook.

A large well connected investor might be able to borrow at broker's call plus about 1%, which might be approximated by T-bills plus 1%. This might be a reasonable estimate for the investor who borrows ($f > 1$). For others the rates are likely to be higher. For instance the prime rate from 1975-1997 varied from a low of 6% (1993) to a high of 19% (1981), according to Associates First Capital Corporation (1998).

As r flunctuates, we expect m to tend to fluctuate inversely (high interest rates tend to depress stock prices for well known reasons). Accordingly, f^* and g_∞ will also fluctuate so the long term S&P index fund investor needs a procedure for periodically re-estimating and revising f^* and his desired level of leverage or cash.

To illustrate the impact of $r_b > r$, where r_b is the investor's borrowing rate, suppose r_b in example (7.2) is $r + 2\%$ or .08, a choice based on the above cited historical values for r, which is intermediate between "good" $r_b \doteq r + 1\%$, and "poor" $r_b \doteq$ the prime rate $\doteq r + 3\%$. We replace r by r_b in equations (7.7) and, if $f^* > 1$, $f^* = 1.3\bar{3}$, $g_\infty(f^*) = .100$, $\text{Sdev}(G_\infty(f^*)) = .20$, $tg_\infty(f^*) = .4k^2$, $t = 4k^2$ years. Note how greatly f^* is reduced.

Comment: Taxes.

Suppose for simplicity that all gains are subject to a constant continuous tax rate T and that all losses are subject to a constant continuous tax refund at the same rate T. Think of the taxing entities, collectively, as a partner that shares a fraction T of all gains and losses. Then equations (7.7) become:

$$\text{(7.7T)} \qquad \begin{aligned} cf^* &= c(m-r)/s^2(1-T) \\ g_\infty(cf^*) &= \left((m-r)^2(c-c^2/2)\right)/s^2 + r(1-T) \\ \text{Sdev}\left(G_\infty(cf^*)\right) &= c(m-r)/s \\ tg_\infty(cf^*) &= k^2c^2/\left(c - c^2/2 + r(1-T)s^2/(m-r^2)\right) \\ t(k,\, cf^*) &= k^2c^2\left((m-r)^2/s^2\right)/\left(\left((m-r)^2/s^2\right)(c-c^2/2) + r(1-T)\right)^2 \end{aligned}$$

It is interesting to see that cf^* increases by the factor $1/(1-T)$. For a high income California resident, the combined state and federal marginal tax rate is 45% so this factor is $1/.55 = 1.82$. The amplification of cf^* leads to the same growth rate as before except for a reduction by rT. The Sdev is unchanged and $t(k,\, cf^*)$ is increased slightly. However, as a practical matter, the much higher leverage needed with a high tax rate is typically not allowed under the margin regulation or is not advisable because the inability to continuously adjust in the real world creates dangers that increase rapidly with the degree of leverage.

(c) **The case for "fractional Kelly"**. Figure 5 shows three g curves for the true $m : m_t = 0.5m_e$, $1.0m_e$ and $1.5m_e$, where m_e is the estimated value of m. The vertical lines and the slanting arrows illustrate the reduction in g for the three choices of: $f = 0.5f_e^*$, f_e^* and $1.5f_e^*$. For example with $f = 0.5f_e^*$ or "half Kelly", we have no loss and achieve the maximum $g = .25$, in case $m_t = 0.5m_e$. But if $m_t = m_e$ then $g = .75$, a loss of .25 and if $m_t = 1.5m_e$ then $g = 1.25$, a loss of 1.0, where all g are in units of $m_e^2/2s^2$. This is indicated both by LOSS_1 and LOSS_2 on the vertical line above $f/f_e^* = .5$, and by the two corresponding arrows which lead upward, and in this case to the right, from this line. A disaster occurs when $m_t = .5m_e$ but we choose $f = 1.5f_e^*$. This combines overbetting f_e^* by 50% with the overestimate of $m_e = 2m_t$. Then $g = -.75$ and we will be ruined. It is still bad to choose $f = f_e^*$ when $m_t = .5m_e$ for then $g = 0$ and we suffer increasingly wild oscillations, both up and down, around our initial capital. During a large downward oscillation experience shows that bettors will generally either quit or be eliminated by a minimum bet size requirement.

Some lessons here are: (1) To the extent $\mathbf{m_e}$ is an uncertain estimate of $\mathbf{m_t}$, it is wise to assume $\mathbf{m_t < m_e}$ and to choose $\mathbf{f < f_e^*}$ by enough to prevent $\mathbf{g \leq 0}$.

Estimates of m_e in the stock market have many uncertainties and, in particular, are more likely to be too high than too low. Securities prices follow a "non-stationary process" where m and s vary somewhat unpredictably over time. The economic situation can change for companies, industries, or the economy as a whole. Systems that worked may be partly or entirely based on data mining so m_t may be substantially less than m_e. Changes in the "rules" such as commissions, tax laws, margin regulations, insider trading laws, etc., can also affect m_t. Systems that do work attract capital, which tends to push exceptional m_t down towards average values. The drift down means $m_e > m_t$ is likely.

Sports betting has much the same caveats as the securities markets, with its own differences in detail. Rules changes, for instance, might include: adding expansion teams; the three point rule in basketball; playing overtime sessions to break a tie; changing types of bats, balls, gloves, racquets or surfaces.

Blackjack differs from the securities and sports betting markets in that the probabilities of outcomes can in principle generally be either calculated or simulated to any desired degree of accuracy. But even here m_t is likely to be at least somewhat less than m_e. Consider player fatigue and errors, calculational errors and mistakes in applying either blackjack theory or Kelly theory (e.g. calculating f^* correctly, for which some of the issues have been discussed aboved), effects of a fixed shuffle point, non-random shuffling, preferential shuffling, cheating, etc.

(2) Subject to (1), choosing f in the range $0.5f_e^* \leq f < f_e^*$ offers protection against $g \leq 0$ with a reduction of g that is likely to be no more than 25%.

Example 7.3. The great compounder. In 1964 a young hedge fund manager acquired a substantial interest in a small New England textile company called Berkshire Hathaway. The stock traded then at 20. In 1998 it traded at 70,000, a multiple of 3500, and an annualized compound growth rate of about 27%, or an instantaneous rate of 24%. The once young hedge fund manager Warren Buffett is now acknowledged as the greatest investor of our time, and the world's second richest man. You may read about Buffett in (Buffett and Clark, 1997), Hagstrom (1994), Kilpatrick (1994), and Lowenstein (1995). If, as I was, you were fortunate enough to meet Buffett and identify the Berkshire opportunity, what strategy does our method suggest? Assume (the somewhat smaller drift rate) $m = .20$, $s = .15$, $r = .06$. (Note: Plausible arguments for a smaller future drift rate include regression towards the mean, the increasing size of Berkshire, and risk from the aging of management. A counter-argument is that Berkshire's compounding rate has been as high in its later years as in its earlier years. However, the S&P 500 Index has performed much better in recent years so the spread between the growth rates of the Index and of Berkshire has been somewhat less. So, if we expect the Index growth rate to revert towards the historical mean, then we expect Berkshire to do so even more. From equations (7.3) or (7.7),

$$f^* = 6.2\bar{2} \qquad g_\infty(f^*) = .49\bar{5} \qquad \mathrm{Sdev}(G_\infty(f^*)) = .9\bar{3}$$

$$tg_\infty(f^*) = 1.76k^2 \qquad t = 3.54k^2 \text{ years}$$

Compare this to the unlevered portfolio, where $f = 1$ and $c = 1/6.2\bar{2} \doteq .1607$. We find:

$$f = 1 \qquad g_\infty(f) = .189 \qquad \mathrm{Sdev}(G_\infty(f)) = .15$$

$$t_k g_\infty(f) = .119k^2 \qquad t_k = 0.63k^2 \text{ years.}$$

Leverage to the level $6.2\bar{2}$ would be inadvisable here in the real world because securities prices may change suddenly and discontinously. In the crash of October, 1987, the S&P 500 index dropped 23% in a single day. If this happened at leverage of 2.0, the new leverage would suddenly be 77/27=2.85 before readjustment by selling part of the portfolio. In the case of Berkshire, which is a large well-diversified portfolio, suppose we chose the conservative $f = 2.0$. Note that this is the maximum initial leverage allowed "customers" under current regulations. Then $g_\infty(2) = 0.295$. The values in 30 years for median V_∞/V_0 are approximately: $f = 1$, $V_\infty/V_0 = 288$; $f = 2$, $V_\infty/V_0 = 6,974$; $f = 6.2\bar{2}$, $V_\infty/V_0 = 2.86 \text{ x } 10^6$. So the differences in results of leveraging are enormous in a generation. (Note: Art Quaife reports $s = .24$ for 1980-1997. The reader is invited to explore the example with this change.)

The results of section 3 apply directly to this continuous approximation model of a (possibly) leveraged securities portfolio. The reason is that both involve the same "dynamics", namely $\log G_n(f)$ is approximated as (scaled) Brownian motion with drift. So we can answer the same questions here for our portfolio that were answered in section 3 for casino betting. For instance (3.2) becomes

$$\text{(7.8)} \qquad \text{Prob}(V(t)/V_0 \le x \text{ for some } t) = x^\wedge(2g_\infty/\text{Var}(G_\infty))$$

where $\wedge$ means exponentiation and $0 < x < 1$. Using (7.4), for $r = 0$ and $f = f^*$, $2g_\infty/\text{Var}(G_\infty) = 1$ so this simplifies to

$$\text{(7.9)} \qquad \text{Prob}(\cdot) = x$$

Compare with Example 3.3. For $0 < r < m$ and $f = f^*$ the exponent of x in (7.9) becomes $1 + 2rs^2/(m-r)^2$ and has a positive first derivative so, as r increases, $P(\cdot)$ decreases (since $0 < x < 1$, tending to 0 as r tends to m, which is what we expect.

(d) **A remarkable formula.**

In earlier versions of this paper the exponent in equations (3.2), (7.8) and (7.9) were off by a factor of 2, which I had inadvertently dropped during my derivation. Subsequently Don Schlesinger posted (without details) two more general continuous approximation formulas for the $r = 0$ case on the internet at www.bjmath.com dated June 19, 1997.

If V_0 is the initial investment and $y > 1 > x > 0$ then for f^* the probability that $V(t)$ reaches yV_0 before xV_0 is

$$\text{(7.10)} \qquad \text{Prob}\left(V(t, f^*) \text{ reaches } yV_0 \text{ before } xV_0\right) = (1-x)/\left(1-(x/y)\right)$$

and more generally, for $f = cf^*$, $0 < c < 2$,

$$\text{(7.11)} \qquad \text{Prob}\left(V(t, cf^*) \text{ reaches } yV_0 \text{ before } xV_0\right)$$

$$= [1 - x^\wedge(2/c - 1)]/[1 - (x/y)^\wedge(2/c - 1)]$$

where $\wedge$ means exponentiation.

Clearly (7.10) follows from (7.11) by choosing $c = 1$. The $r = 0$ case of our equation (7.8) follows from (7.11) and the $r = 0$ case of our equation (7.9) follows from (7.10). We can derive a generalization of (7.11) by using the classical gambler's ruin formula (Cox and Miller, page 31, eqn. (2.0)) and passing to the limit as step size tends to zero (Cox and Miller, pp. 205-6), where we think of $\log(V(t,\, f)/V_0)$ as following a diffusion process with mean g_∞ and variance $v(G_\infty)$, initial value 0, and absorbing barriers at $\log y$ and $\log x$. The result is

(7.12) $\quad \text{Prob}\,(V(t,\, cf^*) \text{ reaches } yV_0 \text{ before } xV_0) = [1 - x^\wedge a]/[1 - (x/y)^\wedge a]$

where $a = 2g_\infty/V(G_\infty) = 2M/V$ where M and V are the drift and variance, respectively, of the diffusion process per unit time. Alternatively, (7.12) is a simple restatement of the known solution for the Wiener process with two absorbing barriers (Cox and Miller, example 5.5).

As Schlesinger notes, choosing $x = 1/2$ and $y = 2$ in (7.10) gives $\text{Prob}(V(t,\, f^*)$ doubles before halving$) = 2/3$. Now consider a gambler or investor who focuses only on values $V_n = 2^n V_0$, $n = 0, \pm 1, \pm 2, \ldots$ multiples of his initial capital. In log space, $\log(V_n/V_0) = n \log 2$ so we have a random walk on the integer multiples of $\log 2$, where the probability of an increase is $p = 2/3$ and of a decrease, $q = 1/3$. This gives us a convenient compact visualization of the Kelly strategy's level of risk.

If instead we choose $c = 1/2$ ("half Kelly"), equation (7.11) gives $\text{Prob}(V(t,\, f^*/2)$ doubles before halving$) = 8/9$ yet the growth rate $g_\infty(f^*/2) = .75 g_\infty(f^*)$ so "half Kelly" has 3/4 the growth rate but much less chance of a big loss.

A second useful visualization of comparative risk comes from equation (7.8) which gives

(7.13) $\quad \text{Prob}\,(V(t,\, cf^*)/V_0 \le x \text{ for some } t) = x^\wedge(2/c - 1)$

For $c = 1$ we had $\text{Prob}(\cdot) = x$ and for $c = 1/2$ we get $\text{Prob}(\cdot) = x^3$. Thus "half Kelly" has a much lessened likelihood of severe capital loss. The chance of ever losing half the starting capital is 1/2 for $f = f^*$ but only 1/8 for $f = f^*/2$. My gambling and investment experience, as well as reports from numerous blackjack players and teams, suggests that most people strongly prefer the increased safety and psychological comfort of "half Kelly" (or some nearby value), in exchange for giving up 1/4 of their growth rate.