is a net gain so we find $f^* < .02$. The greater a is, the more important is the effect of this term so the more we have to reduce f to get f^*, as Table 5.1 clearly shows. When there is a spectrum of favorable situations the solution is more complex and can be found through standard multivariable optimization techniques.

The more complex Case 2 corresponds to what the serious blackjack player is likely to need to do in practice. He will have to limit his current maximum bet to some multiple of his current minimum bet. As his bankroll increases or decreases, the corresponding bet sizes will increase or decrease proportionately.

6 Sports Betting

In 1993 an outstanding young computer science Ph.D. told me about a successful sports betting system that he had developed. Upon review I was convinced. I made suggestions for minor simplifications and improvements. Then we agreed on a field test. We found a person who was extremely likely to always be regarded by the other sports bettors as a novice. I put up a test bankroll of $50,000 and we used the Kelly system to estimate our bet size.

We bet on 101 days in the first four and a half months of 1994. The system works for various sports. The results appear in Figures 3 and 4. After 101 days of bets, our $50,000 bankroll had a profit of $123,000, about $68,000 from Type 1 sports and about $55,000 from Type 2 sports. The expected returns are shown as about $62,000 for Type 1 and about $27,000 for Type 2. One might assign the additional $34,000 actually won to luck. But this is likely to be at most partly true because our expectation estimates from the model were deliberately chosen to be conservative. The reason is that using too large an f^* and overbetting is much more severely penalized than using too small an f^* and underbetting.

Though $123,000 is a modest sum for some, and insignificant by Wall Street standards, the system performed as predicted and passed its test. We were never more than a few thousand behind. The farthest we had to invade our bankroll to place bets was about $10,000.

Our typical expectation was about 6% so our total bets ("action") were about $2,000,000 or about $20,000 per day. We typically placed from five to fifteen bets a day and bets ranged from a few hundred dollars to several thousand each, increasing as our bankroll grew.

Though we had a net win, the net results by casino varied by chance from a substantial loss to a large win. Particularly hard hit was the "sawdust joint" Little Caesar's. It "died" towards the end of our test and I suspect that sports book losses to us may have expedited its departure.

One feature of sports betting which is of interest to Kelly users is the prospect of betting on several games at once. This also arises in blackjack when (a) a player bets on multiple hands or (b) two or more players share a common bankroll. The standard techniques readily solve such problems. We illustrate with:

Example 6.1. Suppose we bet simultaneously on two independent favorable coins with betting fractions f_1 and f_2 and with success probabilities p_1 and p_2, respectively. Then the expected growth rate is given by

$$\begin{aligned} g(f_1, f_2) &= p_1 p_2 ln(1 + f_1 + f_2) + p_1 q_2 ln(1 + f_1 - f_2) \\ &+ q_1 p_2 ln(1 - f_1 + f_2) + q_1 q_2 ln(1 - f_1 - f_2) \end{aligned}$$

To find the optimal f_1^* and f_2^* we solve the simultaneous equations $\partial g/\partial f_1 = 0$ and $\partial g/\partial f_2 = 0$. The result is

$$f_1 + f_2 = \frac{p_1 p_2 - q_1 q_2}{p_1 p_2 + q_1 q_2} \equiv c$$

$$f_1 - f_2 = \frac{p_1 q_2 - q_1 p_2}{p_1 q_2 + q_1 p_2} \equiv d \tag{6.1}$$

$$f_1^* = (c + d)/2 \qquad f_2^* = (c - d)/2$$

These equations pass the symmetry check: interchanging 1 and 2 throughout maps the equation set into itself.

An alternate form is instructive. Let $m_i = p_i - q_i$, $i = 1, 2$ so $p_i = (1 + m_i)/2$ and $q_i = (1 - m_i)/2$. Substituting in (6.1) and simplifying leads to:

$$c = \frac{m_1 + m_2}{1 + m_1 m_2} \qquad d = \frac{m_1 - m_2}{1 - m_1 m_2} \tag{6.2}$$

$$f_1^* = \frac{m_1 (1 - {m_2}^2)}{1 - {m_1}^2 {m_2}^2} \qquad f_2^* = \frac{m_2 (1 - {m_1}^2)}{1 - {m_1}^2 {m_2}^2}$$

which shows clearly the factors by which the f_i^* are each reduced from m_i^*. Since the m_i are typically a few percent, the reduction factors are typically very close to 1.

In the special case $p_1 = p_2 = p$, $d = 0$ and $f^* = f_1^* = f_2^* = c/2 = (p-q)/(2(p^2+q^2))$. Letting $m = p-q$ this may be written $f^* = m/(1+m^2)$ as the optimal fraction to bet on each coin simultaneously, compared to $f^* = m$ to bet on each coin sequentially.

Our simultaneous sports bets were generally on different games and typically not numerous so they were approximately independent and the appropriate fractions were only moderately less than the corresponding single bet fractions. Question: Is this always true for independent simultaneous bets? Simultaneous bets on blackjack hands at different tables are independent but at the same table they have a pairwise correlation that has been estimated at 0.5 (Griffin, 1995, p.142). This should substantially reduce the Kelly fraction per hand. The blackjack literature discusses approximations to these problems. On the other hand, correlations between the returns on securities can range from nearly -1 to nearly 1. An extreme correlation often can be exploited to great advantage through the techniques of "hedging". The risk averse investor may be able to acquire combinations of securities where the expectations add and the risks tend to cancel. The optimal betting fraction may be very large.

The next example is a simple illustration of the important effect of covariance on the optimal betting fraction.

Example 6.2 We have two favorable coins as in the previous example but now their outcomes need not be independent. For simplicity assume the special case where the two bets have the same payoff distributions, but with a joint distribution as in Table 6.1.

Now $c + m + b = (1 + m)/2$ so $b = (1 - m)/2 - c$ and therefore $0 \le c \le (1 - m)/2$.

TABLE 6.1 Joint distribution of two "identical" favorable coins with correlated outcomes.

$X_1:$	$X_2: 1$	-1
1	c+m	b
-1	b	c

Calculation shows $\mathrm{Var}(X_i) = 1 - m^2$, $\mathrm{Cor}(X_1, X_2) = 4c - (1 - m)^2$ and $\mathrm{Cor}(X_1, X_2) = [4c - (1 - m)^2]/(1 - m^2)$. The symmetry of the distribution shows that $g(f_1, f_2)$ will have its maximum at $f_1 = f_2 = f$ so we simply need to maximize $g(f) = (c + m)\ln(1 + 2f) + c\ln(1 - 2f)$. The result is

$f^* = m/(2(2c+m))$. We see that for m fixed, as c decreases from $(1-m)/2$ and $\mathrm{cor}(X_1, X_2) = 1$, to 0 and $\mathrm{cor}(X_1, X_2) = -(1-m)/(1+m)$, f^* for each bet increases from $m/2$ to $1/2$, as in Table 6.2.

TABLE 6.2 f^* increases as Cor (X_1, X_2) decreases.

$\mathrm{Cor}(X_1, X_2)$	c	f^*
1	$(1-m)/2$	$m/2$
0	$(1-m^2)/4$	$m/(1+m^2)$
$-(1-m)/(1+m)$	0	$1/2$

It is important to note that for an exact solution or an arbitrarily accurate numerical approximation to the simultaneous bet problem, covariance or correlation information is not enough. We need to use the entire joint distribution to construct the g function.

We stopped sports betting after our successful test for reasons including: (1) It required a person on site in Nevada. (2) Large amounts of cash and winning tickets had to be transported between casinos. We believed this was very risky. To the sorrow of others, subsequent events confirmed this. (3) It was not economically competitive with our other operations.

If it becomes possible to place bets telephonically from out of state and to transfer the corresponding funds electronically, we may be back.

7 Wall Street: the biggest game

To illustrate both the Kelly criterion and the size of the securities markets, we return to the study of the effects of correlation as in Example 6.2. Consider the more symmetric and esthetically pleasing pair of bets U_1 and U_2, with joint distribution given in Table 7.1

TABLE 7.1 Joint distribution of U_1 and U_2.

$U_1:$	$U_2: m_2+1$	m_2-1
m_1+1	a	$1/2-a$
m_1-1	$1/2-a$	a