TABLE 4.2 The long run: $(g_2 - g_1)/s$ after n trials.

f_1	f_2	$n = 10^4$	$n = 4 * 10^4$	$n = 16 * 10^4$	$n = 10^6$
.01	.02	.5	1.0	2.0	5.0
.03	.02	.5	1.0	2.0	5.0
.03	.01	.000133	.000267	.000534	.001335

The factor $\sqrt{R}$ in Table 4.1 shows how much more slowly f_2 dominates f_1 in Case 2 versus Case 1. The ratio $(g_2 - g_1)/s_2$ is $\sqrt{R}$ times as large so the same level of dominance takes R times as long. When the real world comparisons of strategies for practical reasons often use Case 2 comparisons rather than the more appropriate Case 1 comparisons, the dominance of f^* is further obscured. An example is players with different betting fractions at blackjack. Case 1 corresponds to both betting on the same sequence of hands. Case 2 corresponds to them playing at different tables (not the same table, because Case 2 assumes independence). (Because of the positive correlation between payoffs on hands played at the same table, this is intermediate between Case 1 and Case 2.)

It is important to understand that "the long run", i.e. the time it takes for f^* to dominate a specified neighbor by a specified probability, can vary without limit. Each application requires a separate analysis. In cases such as example 4.1, where dominance is "slow", one might argue that using f^* is not important. As an argument against this, consider two coin-tossing games. In game 1 your edge is 1.0%. In game 2 your edge is 1.1%. With one unit bets, after n trials the difference in expected gain is $E_2 - E_1 = .001n$ with standard deviation s of about $\sqrt{2n}$ hence $(E_2 - E_1)/s \doteq .001\sqrt{n}/\sqrt{2}$ which is 1 when $n = 2 * 10^6$. So it takes two million trials to have an 84% chance of the game 2 results being better than the game 1 results. Does that mean it's unimportant to select the higher expectation game?

5 Blackjack

For a general discussion of blackjack, see Thorp (1962, 1966), Wong (1994) and Griffin (1995). The Kelly criterion was introduced for blackjack by Thorp (1962). The analysis is more complicated than that of coin tossing because the payoffs are not simply one to one. In particular the variance is generally more than 1 and the Kelly fraction tends to be less than for coin tossing with the same expectation. Moreover, the distribution of various payoffs depends on the player advantage. For instance the frequency of pair splitting, doubling down, and blackjacks all vary as the advantage changes. By binning the probability of payoff types according to ex ante expectation, and solving the Kelly equations on a computer, a strategy can be found which is as close to optimal as desired.

There are some conceptual subtleties which are noteworthy. To illustrate them we'll simplify to the coin toss model.

At each trial, we have with probability .5 a "favorable situation" with gain or loss X per unit bet such that $P(X = 1) = .51$, $P(X = -1) = .49$ and with probability .5 an unfavorable situation with gain or loss Y per unit bet such that $P(Y = 1) = .49$ and $P(Y = -1) = .51$. We know before we bet whether X or Y applies.

Suppose the player must make small "waiting" bets on the unfavorable situations in order to be able to exploit the favorable situations. On these he will place "large" bets. We consider two cases.

Case 1. Bet f_0 on unfavorable situations and find the optimal f^* for favorable situations. We have

$$\begin{aligned} g(f) &= .5\,(.51 \log(1+f) + .49 \log(1-f)) \\ &+ .5\,(.49 \log(1+f_0) + .51 \log(1-f_0)) \end{aligned} \tag{5.1}$$

Since the second expression in (5.1) is constant, f maximizes $g(f)$ if it maximizes the first expression, so $f^* = p - q = .02$, as usual. It is easy to verify that when there is a spectrum of favorable situations the same recipe, $f_i^* = p_i - q_i$ for the ith situation, holds. Again, in actual blackjack f_i^* would be adjusted down somewhat for the greater variance. With an additional constraint such as $f_i \leq k f_0$, where k is typically some integral multiple of f_0, representing the betting spread adopted by a prudent player, then the solution is just $f_i \leq \min(f_i^*, k f_0)$.

Curiously, a seemingly similar formulation of the betting problem leads to rather different results.

Case 2. Bet f in favorable situations and af in unfavorable situations, $0 \leq a \leq 1$.

Now the bet sizes in the two situations are linked and both the analysis and results are more complex. We have a Kelly growth rate of

$$\begin{aligned} g(f) &= .5\,(.51 \log(1+f) + .49 \log(1-f)) \\ &+ .5\,(.49 \log(1+af) + .51 \log(1-af)) \end{aligned} \tag{5.2}$$

If we choose $a = 0$ (no bet in unfavorable situations) then the maximum value for $g(f)$ is at $f^* = .02$, the usual Kelly fraction.

If we make "waiting bets", corresponding to some value of $a > 0$, this will shift the value of f^* down, perhaps even to 0. The expected gain divided by the expected bet is $.02(1-a)/(1+a)$, $a \geq 0$. If $a = 0$ we get .02, as expected. If $a = 1$, we get 0, as expected: this is a fair game and the Kelly fraction is $f^* = 0$. As a increases from 0 to 1 the (optimal) Kelly fraction f^* decreases from .02 to 0. Thus the Kelly fraction for favorable situations is less <u>in this case</u> when bets on unfavorable situations reduce the overall advantage of the game.

Arnold Snyder called to my attention the fact that Winston Yamashita had (also) made this point (March 18, 1997) on the "free" pages, miscellaneous section, of Stanford Wong's web site.

For this example, we find the new f^* for a given value of a, $0 < a < 1$, by solving $g'(f) = 0$. A value of $a = 1/3$, for instance, corresponds to a bet of 1/3 unit on Y and 1 unit on X, a betting range of 3 to 1. The overall expectation is .01. Calcuation shows $f^* = .012001$. Table 5.1 shows how f^* varies with a.

Table 5.1 f^* **versus** a.

a	f^*	a	f^*	a	f^*
0	.0200	1/3	.0120	.7	.0040
.1	.0178	.4	.0103	.8	.0024
.2	.0154	.5	.0080	.9	.0011
.3	.0128	.6	.0059	1.0	.0000

To understand why Case 1 and Case 2 have different f^*, look first at equation (5.1). The part of $g(f)$ corresponding to the unfavorable situations is fixed when f_0 is fixed. Only the part of $g(f)$ corresponding to the favorable situations is affected by varying f. Thus we maximize $g(f)$ by maximizing it over just the favorable situations. Whatever the result, it is then reduced by a fixed quantity, the part of g containing f_0. On the other hand, in equation (5.2) both parts of $g(f)$ are affected when f varies, because the fraction af used for unfavorable situations bears the constant ratio a to the fraction f used in favorable situations. Now the first term, for the favorable situations, has a maximum at $f = .02$, and is approximately "flat" nearby. But the second term, for the unfavorable situations, is negative and decreasing moderately rapidly at $f = .02$. Therefore, it we reduce f somewhat, this term increases somewhat, while the first term decreases only very slightly. There is a net gain so we find $f^* < .02$. The greater a is, the more important is the effect of this term so the more we have to reduce f to get f^*, as Table 5.1 clearly shows. When there is a spectrum of favorable situations the solution is more complex and can be found through standard multivariable optimization techniques.

The more complex Case 2 corresponds to what the serious blackjack player is likely to need to do in practice. He will have to limit his current maximum bet to some multiple of his current minimum bet. As his bankroll increases or decreases, the corresponding bet sizes will increase or decrease proportionately.

6 Sports Betting

In 1993 an outstanding young computer science Ph.D. told me about a successful sports betting system that he had developed. Upon review I was convinced. I made suggestions for minor simplifications and improvements. Then we agreed on a field test. We found a person who was extremely likely to always be regarded by the other sports bettors as a novice. I put up a test bankroll of $50,000 and we used the Kelly system to estimate our bet size.

We bet on 101 days in the first four and a half months of 1994. The system works for various sports. The results appear in Figures 3 and 4. After 101 days of bets, our $50,000 bankroll had a profit of $123,000, about $68,000 from Type 1 sports and about $55,000 from Type 2 sports. The expected returns are shown as about $62,000 for Type 1 and about $27,000 for Type 2. One might assign the additional $34,000 actually won to luck. But this is likely to be at most partly true because our expectation estimates from the model were deliberately chosen to be conservative. The reason is that using too large an f^* and overbetting is much more severely penalized than using too small an f^* and underbetting.

Though $123,000 is a modest sum for some, and insignificant by Wall Street standards, the system performed as predicted and passed its test. We were never more than a few thousand behind. The farthest we had to invade our bankroll to place bets was about $10,000.

Our typical expectation was about 6% so our total bets ("action") were about $2,000,000 or about $20,000 per day. We typically placed from five to fifteen bets a day and bets ranged from a few hundred dollars to several thousand each, increasing as our bankroll grew.