$f > f^*$ beats Kelly with probability $p > 1/2$. If instead $n = 2$, $f > f^*$ wins with probability p^2 and $p^2 > 1/2$ if $p > 1/\sqrt{2} \doteq .7071$. Also, $f < f^*$ wins with probability $1 - p^2$ and $1 - p^2 > 1/2$ if $p^2 < 1/2$, i.e. $p < 1/\sqrt{2} = .7071$. So when $n = 2$, Kelly always loses more than half the time to some other f unless $p = 1/\sqrt{2}$.

We now have the formulas we need to explore many practical applications of the Kelly criterion.

4 The Long Run: When Will The Kelly Strategy "Dominate"?

The late John Leib wrote several articles for Blackjack Forum which were critical of the Kelly criterion. He was much bemused by "the long run". What is it and when, if ever, does it happen?

We begin with an example.

Example 4.1 $\qquad p = .51, \quad n = 10,000$

V_i and s_i, $i = 1, 2$ are the variance and standard deviation, respectively, for 3(e) Cases 1 and 2, and $R = V_2/V_1 = (a^2 + b^2)/(a - b)^2$ so $s_2 = s_1\sqrt{R}$. Table 4.1 summarizes some results. We can also approximate $\sqrt{R}$ with a power series estimate using only the first term of a and of b: $a \doteq 2f_1, b \doteq 2f_2$ so $\sqrt{R} \doteq \sqrt{f_1^2 + f_2^2}/\mid f_1 - f_2 \mid$. The approximate results, which agree extremely well, are 2.236, 3.606 and 1.581, respectively.

TABLE 4.1 Comparing strategies

f_1	f_2	$g_2 - g_1$	s_1	$(g_2 - g_1)/s_1$	$\sqrt{R}$
.01	.02	.00005001	.00010000	.50	2.236
.03	.02	.00005004	.00010004	.50	3.604
.03	.01	.00000003	.00020005	.00013346	1.581

The first two rows show how nearly symmetric the behavior is on each side of the optimal $f^* = .02$. The column $(g_2 - g_1)/s_1$ shows us that $f^* = .02$ only has a .5 standard deviation advantage over its neighbors $f = .01$ and $f = .03$ after $n = 10,000$ trials. Since this advantage is proportional to $\sqrt{n}$, the column $(g_2 - g_1)/s_1$ from Table 4.1 gives the results of Table 4.2:

TABLE 4.2 The long run: $(g_2 - g_1)/s$ after n trials.

f_1	f_2	$n = 10^4$	$n = 4 * 10^4$	$n = 16 * 10^4$	$n = 10^6$
.01	.02	.5	1.0	2.0	5.0
.03	.02	.5	1.0	2.0	5.0
.03	.01	.000133	.000267	.000534	.001335

The factor $\sqrt{R}$ in Table 4.1 shows how much more slowly f_2 dominates f_1 in Case 2 versus Case 1. The ratio $(g_2 - g_1)/s_2$ is $\sqrt{R}$ times as large so the same level of dominance takes R times as long. When the real world comparisons of strategies for practical reasons often use Case 2 comparisons rather than the more appropriate Case 1 comparisons, the dominance of f^* is further obscured. An example is players with different betting fractions at blackjack. Case 1 corresponds to both betting on the same sequence of hands. Case 2 corresponds to them playing at different tables (not the same table, because Case 2 assumes independence). (Because of the positive correlation between payoffs on hands played at the same table, this is intermediate between Case 1 and Case 2.)

It is important to understand that "the long run", i.e. the time it takes for f^* to dominate a specified neighbor by a specified probability, can vary without limit. Each application requires a separate analysis. In cases such as example 4.1, where dominance is "slow", one might argue that using f^* is not important. As an argument against this, consider two coin-tossing games. In game 1 your edge is 1.0%. In game 2 your edge is 1.1%. With one unit bets, after n trials the difference in expected gain is $E_2 - E_1 = .001n$ with standard deviation s of about $\sqrt{2n}$ hence $(E_2 - E_1)/s \doteq .001\sqrt{n}/\sqrt{2}$ which is 1 when $n = 2 * 10^6$. So it takes two million trials to have an 84% chance of the game 2 results being better than the game 1 results. Does that mean it's unimportant to select the higher expectation game?

5 Blackjack

For a general discussion of blackjack, see Thorp (1962, 1966), Wong (1994) and Griffin (1995). The Kelly criterion was introduced for blackjack by Thorp (1962). The analysis is more complicated than that of coin tossing because the payoffs are not simply one to one. In particular the variance is generally more than 1 and the Kelly fraction tends to be less than for coin tossing