bet a fraction of your current capital equal to your expectation. This is modified somewhat in practice (generally down) to allow for having to make some negative expectation "waiting bets", for the higher variance due to the occurrence of payoffs greater than one to one, and when more than one hand is played at a time.

Here are the properties that made the Kelly criterion so appealing. For ease of understanding, we illustrate using the simplest case, coin tossing, but the concepts and conclusions generalize greatly.

2 Coin Tossing

Imagine that we are faced with an infinitely wealthy oppponent who will wager even money bets made on repeated independent trials of a biased coin. Further, suppose that on each trial our win probability is $p > 1/2$ and the probability of losing is $q = 1 - p$. Our initial capital is X_0. Suppose we choose the goal of maximizing the expected value $E(X_n)$ after n trials. How much should we bet, B_k, on the kth trial? Letting $T_k = 1$ if the kth trial is a win and $T_k = -1$ if it is a loss, then $X_k = X_{k-1} + T_k B_k$ for $k = 1, 2, 3, \ldots$, and $X_n = X_0 + \sum_{k=1}^{n} T_k B_k$. Then

$$E(X_n) = X_0 + \sum_{k=1}^{n} E(B_k T_k) = X_0 + \sum_{k=1}^{n} (p - q) E(B_k).$$

Since the game has a positive expectation, i.e., $p - q > 0$ in this even payoff situation, then in order to maximize $E(X_n)$ we would want to maximize $E(B_k)$ at each trial. Thus, to maximize expected gain we should bet *all of our resources* at each trial. Thus $B_1 = X_0$ and if we win the first bet, $B_2 = 2X_0$, etc. However, the probability of ruin is given by $1 - p^n$ and with $p < 1$, $\lim_{n \to \infty} [1 - p^n] = 1$ so ruin is almost sure. Thus the "bold" criterion of betting to maximize expected gain is usually undesirable.

Likewise, if we play to minimize the probability of eventual ruin (i.e., "ruin" occurs if $X_k = 0$ on the kth outcome) the well-known gambler's ruin formula in Feller (1966) shows that we minimize ruin by making a *minimum* bet on each trial, but this unfortunately also minimizes the expected gain. Thus "timid" betting is also unattractive.

This suggests an intermediate strategy which is somewhere between maximizing $E(X_n)$ (and assuring ruin) and minimizing the probability of ruin (and minimizing $E(X_n)$). An asymptotically optimal strategy was first proposed by J.L. Kelly (1956).

In the coin-tossing game just described, since the probabilities and payoffs for each bet are the same, it seems plausible that an "optimal" strategy will involve always wagering the same fraction f of your bankroll. To make this possible we shall assume from here on that capital is infinitely divisible. This assumption usually does not matter much in the interesting practical applications.

If we bet according to $B_i = fX_{i-1}$, where $0 \leq f \leq 1$, this is sometimes called "fixed fraction" betting. Where S and F are the number of successes and failures, respectively, in n trials, then our capital after n trials is $X_n = X_0(1+f)^S(1-f)^F$, where $S + F = n$. With f in the interval $0 < f < 1$, $\Pr(X_n = 0) = 0$. Thus "ruin" in the technical sense of the gambler's ruin problem cannot occur. "Ruin" shall henceforth be reinterpreted to mean that for arbitrarily small positive ε, $\lim_{n\to\infty}[\Pr(X_n \leq \varepsilon)] = 1$. Even in this sense, as we shall see, ruin *can* occur under certain circumstances.

We note that since

$$e^{n \log \left[\frac{X_n}{X_0}\right]^{1/n}} = \frac{X_n}{X_0},$$

the quantity

$$G_n(f) = \log \left[\frac{X_n}{X_0}\right]^{1/n} = \frac{S}{n}\log(1+f) + \frac{F}{n}\log(1-f)$$

measures the exponential rate of increase per trial. Kelly chose to maximize the expected value of the growth rate coefficient, $g(f)$, where

$$\begin{aligned} g(f) &= E\left\{\log\left[\frac{X_n}{X_0}\right]^{1/n}\right\} = E\left\{\frac{S}{n}\log(1+f) + \frac{F}{n}\log(1-f)\right\} \\ &= p\log(1+f) + q\log(1-f). \end{aligned}$$

Note that $g(f) = (1/n)E(\log X_n) - (1/n)\log X_0$ so for n fixed, maximizing $g(f)$ is the same as maximizing $E\log X_n$. We usually will talk about maximizing $g(f)$ in the discussion below. Note that

$$g'(f) = \frac{p}{1+f} - \frac{q}{1-f} = \frac{p-q-f}{(1+f)(1-f)} = 0$$

when $f = f^* = p - q$.

Now

$$g''(f) = -p/(1+f)^2 - q/(1-f)^2 < 0$$

so that $g'(f)$ is monotone strictly decreasing on $[0, 1)$. Also $g'(0) = p - q > 0$ and $\lim_{f\to 1^-} g'(f) = -\infty$. Therefore by the continuity of $g'(f)$, $g(f)$ has a unique maximum at $f = f^*$, where $g(f^*) = p\log p + q\log q + \log 2 > 0$ Moreover, $g(0) = 0$ and $\lim_{f\to q^-} g(f) = -\infty$ so there is a unique number $f_c > 0$, where $0 < f^* < f_c < 1$, such that $g(f_c) = 0$. The nature of the function $g(f)$ is now apparent and a graph of $g(f)$ versus f appears as shown in Figure 1.

The following theorem recounts the important advantages of maximizing $g(f)$. The details are omitted here but proofs of (i), (ii), (iii), and (vi) for the simple binomial case can be found in Thorp (1969); more general proofs of these and of (iv) and (v) are in Breiman (1961).

Theorem 1 *(i) If* $g(f) > 0$, *then* $\lim_{n\to\infty} X_n = \infty$ *almost surely, i.e., for each* M, $\Pr[\liminf_{n\to\infty} X_n > M] = 1$;

(ii) If $g(f) < 0$, *then* $\lim_{n\to\infty} X_n = 0$ *almost surely; i.e., for each* $\varepsilon > 0$, $\Pr[\limsup_{n\to\infty} X_n < \varepsilon] = 1$;

(iii) If $g(f) = 0$, *then* $\limsup_{n\to\infty} X_n = \infty\, a.s.$ *and* $\liminf_{n\to\infty} X_n = 0\, a.s.$

(iv) Given a strategy Φ^* *which maximizes* $E\log X_n$ *and any other "essentially different" strategy* Φ *(not necessarily a fixed fractional betting strategy), then* $\lim_{n\to\infty} X_n(\Phi^*)/X_n(\Phi) = \infty\, a.s.$

(v) The expected time for the current capital X_n *to reach any fixed preassigned goal* C *is, asymptotically, least with a strategy which maximizes* $E\log X_n$.

(vi) Suppose the return on one unit bet on the ith *trial is the binomial random variable* U_i; *further, suppose that the probability of success is* p_i, *where* $1/2 < p_i < 1$. *Then* $E\log X_n$ *is maximized by choosing on each trial the fraction* $f_i^* = p_i - q_i$ *which maximizes* $E\log(1 + f_i U_i)$.

Part (i) shows that, except for a finite number of terms, the player's fortune X_n will exceed any fixed bound M when f is chosen in the interval $(0, f_c)$. But, if $f > f_c$, part (ii) shows that ruin is almost sure. Part (iii) demonstrates that if $f = f_c$, X_n will (almost surely) oscillate randomly between 0 and $+\infty$. Thus, one author's statement that $X_n \to X_0$ as $n \to \infty$, when $f = f_e$, is clearly contradicted. Parts (iv) and (v) show that the Kelly strategy of maximizing $E\log X_n$ is asymptotically optimal by two important criteria. An "essentially different" strategy is one such that the difference $E \ln X_n^* - E \ln X_n$ between the Kelly strategy and the other strategy grows faster than the standard deviation of $\ln X_n^* - \ln X_n$, ensuring $P(\ln X_n^* - \ln X_n > 0) \to 1$. Part (vi) establishes the validity of utilizing the Kelly method of choosing f_i^* on each trial (even if the probabilities change from one trial to the next) in order to maximize $E \log X_n$.

Example 2.1 Player A plays against an infinitely wealthy adversary. Player A wins even money on successive independent flips of a biased coin with a win probability of $p = .53$ (no ties). Player A has an initial capital of X_0 and *capital is infinitely divisible.* Applying Theorem 1(vi), $f^* = p - q = .53 - .47 = .06$. Thus 6% of current capital should be wagered on each play in order to cause X_n to grow at the fastest rate possible consistent with zero probability of ever going broke. If Player A continually bets a fraction smaller than 6%, X_n will also grow to infinity but the rate will be slower.

If Player A repeatedly bets a fraction larger than 6%, up to the value f_c, the same thing applies. Solving the equation $g(f) = .53 \log(1 + f) + .47 \log(1 - f) = 0$ numerically on a computer yields $f_c = .11973^-$. So, if the fraction wagered is more than about 12%, then even though Player A may temporarily experience the pleasure of a faster win rate, eventual downward fluctuations will inexorably drive the values of X_n toward zero. Calculation yields a growth coefficient of $g(f^*) = f(.06) = .001801$ so that after n successive bets the log of Player A's average bankroll will tend to $.001801n$ times as much money as he started with. Setting $.001801n = \log 2$ gives an expected time of about $n = 385$ to double the bankroll.

The Kelly criterion can easily be extended to uneven payoff games. Suppose Player A wins b units for every unit wager. Further, suppose that on each trial the win probability is $p > 0$ and $pb - q > 0$ so the game is advantageous to Player A. Methods similar to those already described can be used to maximize

$$g(f) = E \log(X_n/X_0) = p \log(1 + bf) + q \log(1 - f).$$

Arguments using calculus yield $f^* = (bp - q)/b$, the optimal fraction of current capital which should be wagered on each play in order to maximize the growth coefficient $g(f)$.

This formula for f^* appeared in Thorp (1984) and was the subject of an April 1997 discussion on the internet at Stanford Wong's website, http://bj21.com (miscellaneous free pages section). One claim was that one can only lose the amount bet so there was no reason to consider the (simple) generalization of this formula to the situation where a unit wager wins b with probability $p > 0$ and loses a with probability q. Then if the expectation $m \equiv bp - aq > 0$, $f^* > 0$ and $f^* = m/ab$. The generalization does stand up to the objection. One can buy on credit in the financial markets and lose much more than the amount bet. Consider buying commodity futures or selling short a security (where the loss is potentially unlimited). See, e.g., Thorp and Kassouf (1967) for an account of the E.L. Bruce short squeeze.

For purists who insist that these payoffs are not binary, consider selling short a binary digital option. These options are described in Browne (1996).

A criticism sometimes applied to the Kelly strategy is that capital is not, in fact, infinitely divisible. In the real world, bets are multiples of a minimum unit, such as \$1 or \$.01 (penny "slots"). In the securities markets, with computerized records, the minimum unit can be as small as desired. With a minimum allowed bet, "ruin" in the standard sense is always possible. It is not difficult to show, however, (see Thorp and Walden, 1966b) that if the minimum bet allowed is small relative to the gambler's initial capital, then the probability of ruin in the standard sense is "negligible" and also that the theory herein described is a useful approximation. This section follows Rotando and Thorp (1992).

3 Optimal growth: Kelly criterion formulas for practitioners

Since the Kelly criterion asymptotically maximizes the expected growth rate of wealth, it is often called the optimal growth strategy. It is interesting to compare it with the other fixed fraction strategies. I will present some results that I have found useful in practice. My object is to do so in a way that is simple and easily understood. These results have come mostly from sitting and thinking about "interesting questions". I have not made a thorough literature search but I know that some of these results have been previously published and in greater mathematical generality. See e.g. Browne (1995, 1996) and the references therein.

(a) The probability of reaching a fixed goal on or before n trials. We first assume coin tossing. We begin by noting a related result for standard Brownian motion. Howard Tucker showed me this in 1974 and it is probably the most useful single fact I know for dealing with diverse problems