APPENDIX II. Derivation of Formula (3.1)

This is based on a note from Howard Tucker. Any errors are mine.
*From the paper by Paranjape & Park, if $X(t)$ is standard Brownian motion, if $a \neq 0$, $b > 0$,

$$P(X(t) \leq at + b,\, 0 \leq t \leq T \mid X(T) = s)$$

$$= \begin{cases} 1 - exp\left\{ -\frac{2b}{T}(aT + B - s) \right\} & \text{if } s \leq aT + b \\ 0 \,\text{if}\; s > aT + B. \end{cases}$$

Write this as:

$$P(X(t) \leq at + b,\, 0 \leq t \leq T \mid X(T)) \overset{a.s.}{=}$$

$$= (1 - exp\{-2b(aT + b - X(T))\frac{1}{T}\} \quad \text{if} \quad X(T) \leq aT + b.$$

Taking expectations of both sides of the above, we get

$$P(X(t) \leq at + b,\, 0 \leq t \leq T)$$

$$= \int_{-\infty}^{aT+b} \left(1 - e^{-2b(aT+b-s)1/T}\right) \frac{1}{\sqrt{2\pi T}} e^{-s^2/2T} ds$$

$$= \frac{1}{\sqrt{2\pi T}} \int_{-\infty}^{aT+b} e^{-s^2/2T} ds - \frac{e^{-2ab}}{\sqrt{2\pi T}} \int_{-\infty}^{aT+b} e^{-(s-2b)^2/2T} ds$$

Hence

(I) $\quad P(X \text{ going above line}\, at + b \,\text{during}\, [O, T]) = 1 -$ previous probability

$$= \frac{1}{\sqrt{2\pi T}} \int_{aT+b}^{\infty} e^{-s^2/2T} ds + e^{-2ab} \cdot \frac{1}{\sqrt{2\pi T}} \int_{-\infty}^{aT-b} e^{-u^2/2T} du, \text{ where}\, u = s - 2b.$$

Now, when $a = 0$, $b > 0$,

$$P\left[\sup_{0 \leq t \leq T} X(t) \geq b\right] = \sqrt{\tfrac{2}{\pi T}} \int_{b}^{\infty} e^{-v^2/2T} dv,$$

which agrees with a known formula (see, e.g., page 261 of Tucker [1967]). When $T \to \infty$, since $\sqrt{T}/T \to 0$ and $\sqrt{T} = s.d.$ of $X(T)$, the first integral $\to 0$, the second integral $\to 1$, and $P(X$ ever rises above line $at + b) = e^{-2ab}$.

*The theorem it comes from is due to Sten Malmquist: On Certain Confidence Countours for Distribution Functions, Ann. Math. Stat., 25 (1954), pp 523-533. This theorem is stated in S.R. Paranjape and C. Park: Distribution of the Supremum of the Two-Parameter Yeh-Wiener Process on the Boundary, J. Appl. Prob. Vol. 10 (1973).

Letting $\propto = a\sqrt{T}$, $\beta = b/\sqrt{T}$, formula I becomes

$$P(\cdot) = N(-\propto - \beta) + e^{-2\propto\beta}N(\propto - \beta) \text{ where } \propto, \beta > 0 \quad \text{or}$$

$$\text{(II)} \quad P(X(t) \leq at + b,\ 0 \leq t \leq T) = 1 - P(\cdot) = N(\propto + \beta) - e^{-2\propto\beta}N(\propto - \beta)$$

for the probability the line is never surpassed. This follows from:

$$\frac{1}{\sqrt{2\pi T}} \int_{aT+b}^{\infty} e^{-s^2/2T} ds = \frac{1}{\sqrt{2\pi}} \int_{a\sqrt{T}+b/\sqrt{T}}^{\infty} e^{-x^2/2} dx = N(-\propto - \beta)$$

$$\text{and} \qquad \frac{1}{\sqrt{2\pi T}} \int_{-\infty}^{aT-b} e^{-u^2/2T} du = N(\propto - \beta)$$

where $s = aT + b$ and $x = s/\sqrt{T} = a\sqrt{T} + b/\sqrt{T}$
and $\propto = a\sqrt{T}$ and $\beta = b/\sqrt{T}$.

The formula becomes:

$$P(\sup[X(t) - (at + b)] \geq 0 : 0 \leq t \leq T)$$

$$\begin{aligned} &= N(-\propto - \beta) + e^{-2ab}N(\propto - \beta) \\ &= N(-\propto - \beta) + e^{-2\propto\beta}N(\propto - \beta) \quad \propto, \beta > 0 \end{aligned}$$

Observe that $P(\cdot) < N(-\propto - \beta) + N(\propto - \beta)$

$$\begin{aligned} &= \{1 - N(\propto + \beta)\} + N(\propto - \beta) \\ &= \int_{-\infty}^{\propto - \beta} \propto (x) dx + \int_{\propto + \beta}^{\infty} \propto (x) dx < 1 \end{aligned}$$

as it should be.