APPENDIX I. Integrals for deriving moments of E_∞

$$I_0\left(a^2,b^2\right) = \int_0^\infty \exp\left[-\left(a^2x^2+b^2/x^2\right)\right]dx$$
$$I_n\left(a^2,b^2\right) = \int_0^\infty x^n \exp\left[-\left(a^2x^2+b^2/x^2\right)\right]dx$$

Given I_0 find I_2

$$\begin{aligned} I_0\left(a^2,b^2\right) &= \int_0^\infty \exp\left[-\left(a^2x^2+b^2/x^2\right)\right]dx \\ &= -\int_\infty^0 \exp\left[-\left(a^2/u^2+b^2u^2\right)\right]\left(-du/u^2\right) \end{aligned}$$

where $x = 1/u$ and $dx = -du/u^2$ so

$$\begin{aligned} I_0\left(a^2,b^2\right) &= \int_0^\infty x^{-2} \exp\left[-\left(b^2x^2+a^2/x^2\right)\right] \\ &= I_{-2}\left(b^2,a^2\right) \end{aligned}$$

(A1) $$\text{hence} \quad I_{-2}\left(a^2,b^2\right) = I_0\left(b^2,a^2\right) = \frac{\sqrt{\pi}}{2\mid b\mid}e^{-2|ab|}$$

$$I_0 = \int_0^\infty \exp\left[-\left(a^2x^2+b^2/x^2\right)\right]dx = U\cdot V\Big|_0^\infty - \int_0^\infty VdU$$

where $U = \exp[\,\cdot\,]$, $dV = dx$, $dU = (\exp[\,\cdot\,])(-2a^2x+2b^2x^{-3})$ and $V = x$ so

$$\begin{aligned} I_0 &= \exp\left[-\left(a^2x^2+b^2/x^2\right)\right]\cdot x\Big|_0^\infty - \int_0^\infty \left(-2a^2x^2+2b^2/x^2\right)\exp\left[-\left(a^2x^2+b^2/x^2\right)\right]dx \\ &= 2a^2I_2\left(a^2,b^2\right) - 2b^2I_{-2}\left(a^2,b^2\right) \end{aligned}$$

Hence:

$$I_0\left(a^2,b^2\right) = 2a^2I_2\left(a^2,b^2\right) - 2b^2I_{-2}\left(a^2,b^2\right)$$

and $I_{-2}(a^2, b^2) = I_0(b^2, a^2)$ so substituting and solving for I_2 gives

$$I_2\left(a^2, b^2\right) = \frac{1}{2a^2}\left\{I_0\left(a^2, b^2\right) + 2b^2 I_0\left(b^2, a^2\right)\right\}$$

Comments.

1. We can solve for all even n by using I_0, I_{-2} and I_2, and integration by parts.

2. We can use the indefinite integral J_0 corresponding to I_0, and the previous methods, to solve for J_{-2}, J_2, and then for all even n. Since

$$\begin{aligned}
I_0\left(a^2, b^2\right) &= \frac{\sqrt{\pi}}{2\mid a\mid}e^{-2|ab|} \qquad \text{then} \\
I_{-2}\left(b^2, a^2\right) &= \frac{\sqrt{\pi}}{2\mid b\mid}e^{-2|ab|} \qquad \text{and} \\
I_2\left(a^2, b^2\right) &= \frac{1}{2a^2}\left\{\frac{\sqrt{\pi}}{2\mid a\mid} + 2b^2\frac{\sqrt{\pi}}{2\mid b\mid}\right\}e^{-2|ab|} \\
&= \frac{\sqrt{\pi}}{4a^2}e^{-2|ab|}\left\{1/\mid a\mid + 2\mid b\mid\right\}
\end{aligned}$$