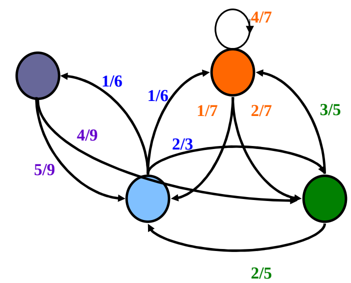
Mathematics for Computer Science

MIT 6.042J/18.062J

Random Walks

Graph with probable transitions



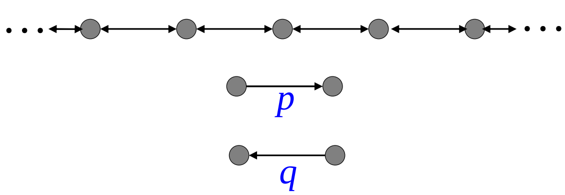
Graph with probable transitions

Questions
- Pr{blue reaches orange *before* green}
- Pr{blue *ever* reaches orange}
- E[#steps blue to orange]
- Average fraction of time at blue

Random Walks

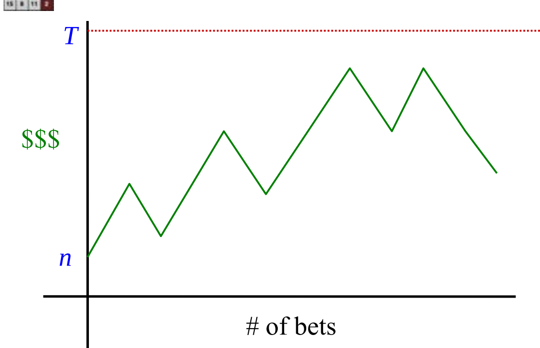
Applications
- Finance – Stocks, options
- Algorithms – web search, clustering
- Physics – Brownian Motion

1-Dimensional Walk



Gambler's Ruin

Gambler's Ruin



The Gambler's Ruin

Parameters:

- n ::= initial capital (stake)
- T ::= gambler's Target
- p ::= Pr{win \$1 bet}
- $q ::= 1 - p$
- m ::= intended profit = $T - n$

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The Gambler's Ruin

Three general cases:

- Biased against $p < 1/2$
- Biased in favor $p > 1/2$
- Unbiased (Fair) $p = 1/2$

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Unbiased Case: $p = q = 1/2$

Let w ::= Pr{reach Target}

$$E[\$\$] = w\cdot(T - n) + (1 - w)\cdot(-n)$$
$$= wT - n$$

But game is *fair*, so E[\$\$ won] = 0

$$w = \frac{n}{T}$$

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Unbiased Case

Consequences

n=500, T=600

Pr{win \$100} = 500/600 ≈ 0.83

n=1,000,000, T=1,000,100

Pr{win \$100} ≈ 0.9999

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Unbiased Case -- More analysis

Wait! Why is

E[\$\$ won] = 0?

Define Random Variables, G_i

$$G_i ::= \begin{cases} 0 & \text{if game ends in } < i \text{ bets} \\ 1 & \text{if gambler wins } i^{\text{th}} \text{ bet} \\ -1 & \text{if gambler loses } i^{\text{th}} \text{ bet} \end{cases}$$

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Infinite Additivity of Expectation

$$\$\$ \text{ won} = \sum_{i=1}^{\infty} G_i$$

$$E[\$\$ \text{ won}] = E[\textstyle\sum G_i]$$
$$= \textstyle\sum E[G_i]$$
$$= \textstyle\sum 0$$
$$= 0$$

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Infinite Additivity?

WAIT!  ALARM!

This is just like
the bet-doubling (St. Petersburg) paradox
(a fair game with guaranteed \$10 win)

Infinite Additivity?

We must verify that

$$\sum_{i=1}^{\infty} \mathrm{E}\left[\left|G_i\right|\right]$$

converges.

Infinite Additivity?

$$|G_i| = \begin{cases} 1 & \text{if} \geq i \text{ bets are made,} \\ 0 & \text{game ends before } i \text{ bets.} \end{cases}$$

$$\mathrm{E}[|G_i|] = \Pr\{\geq i \text{ bets}\}$$

Convergence Condition is Met

In-class Problem 1:
$\Pr\{\text{game takes} \geq i \text{ bets}\} \leq cr^i$
for some $c > 0$, $r < 1$, so

$$\sum_{i=1}^{\infty} \mathrm{E}[|G_i|] \leq c \sum_{i=1}^{\infty} r^i < \infty$$

The End is Certain

$\Pr\{\text{game takes} \geq i \text{ bets}\} \leq cr^i$
so
$\Pr\{\text{game takes forever}\} = 0.$
Already was assumed in:
$\Pr\{\text{loss}\} = 1 - w = 1 - \Pr\{\text{win}\}$

Biased Against: $p < 1/2 < q$

Betting red in US roulette

$$p = 18/38 = 9/19 < 1/2$$

Biased Against: $p < 1/2 < q$

Astonishing Fact!

Pr{win \$100 starting with \$500}

$< 1/37{,}000$!

(was 5/6 in the unbiased case.)

Biased Against: $p < 1/2 < q$

More amazing still!

Pr{win \$100 starting with \$1M}

$< 1/37{,}000$

Pr{win \$100 starting w/ any \$$n$ stake}

$< 1/37{,}000$

Winning in the Biased Case

w_n ::= Pr{win with stake n}

$$w_n = pw_{n+1} + qw_{n-1}$$

$w_0 = 0$ (*Gambler starts broke*)

$w_T = 1$ (*Gambler starts a winner*)

$$w_{n+1} = (1/p)w_n - (q/p)w_{n-1}$$

Winning in the Biased Case

$$w_{n+1} - (1/p)w_n + (q/p)w_{n-1} = 0$$

A linear recurrence: Guess that

$w_n = c^n$ for some c, so

$$c^{n+1} - (1/p)c^n - (q/p)c^{n-1} = 0$$

Winning in the Biased Case

$$c^2 - (1/p)c - (q/p) = 0$$

roots = 1, q/p so

$w_n = (q/p)^n$ and $w_n = 1^n$ satisfy

$$w_{n+1} - (1/p)w_n + (q/p)w_{n-1} = 0$$

Winning in the Biased Case

so $a\,(q/p)^n + b1^n$

satisfies the recurrence. Use boundary conditions at $n = 0,T$ to solve for a and b, and get:

Winning in the Biased Case

$$w_n = \frac{(q/p)^n - 1}{(q/p)^T - 1}$$

for $p \neq q$

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Winning in the Unfair Case

Punchline: for $p < q$:

$$w_n \leq \frac{(q/p)^n}{(q/p)^T} = \left(\frac{p}{q}\right)^m$$

where m ::= $T-n$ = intended profit

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Winning in the Unfair Case

for $p < q$:

$$\left(\frac{p}{q}\right)^m$$

is exponentially decreasing in m,

the intended profit.

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Losing in Roulette

$p = 18/38$, $q = 20/38$

$$\Pr\{\text{win } \$100\} = \left(\frac{18/38}{20/38}\right)^{100} = \left(\frac{9}{10}\right)^{100} < \frac{1}{37,648}$$

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Losing in Roulette

$$\Pr\{\text{win } \$200\} = (\Pr\{\text{win } \$100\})^2 = \left(\frac{1}{37,648}\right)^2 < \frac{1}{70,000,000}$$

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How Many Bets?

What is the expected number of bets for the game to end?

– either by winning \$(T-n) or by going broke (losing \$$n$).

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How Many Bets? Biased Case

E[\$ per bet] = $p - q = 2p-1$
so by Wald's Thm
E[\$ won] = $(2p-1)$ E[# bets]

$$\text{E[\# bets]} = \frac{\text{E[\$ won]}}{(2p-1)}$$

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How Many Bets? Biased Case

But
E[\$ won] = $w_n(T-n) - (1-w_n)n$
so

$$\text{E[\# bets]} = \frac{w_n T - n}{2p-1}$$

for $p \neq 1/2$.

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How Many Bets? Fair Case

E[# bets] = $n(T-n)$ =
(initial stake)·(intended profit)
proof by
- $\lim_{p \to 1/2}$ E[# unfair bets], or
- solving linear recurrence:

$$e_n = p(1 + e_{n+1}) + q(1 + e_{n-1})$$

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Unbiased Case for $T = \infty$

$$pr\{win\} = \frac{n}{T}$$

$$pr\{lose\} = \left(1 - \frac{n}{T}\right) \to 1$$

as $T \to \infty$

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Unbiased Case for $T = \infty$

If you lose a play aiming for goal T_1, then you would lose for $T_2 > T_1$, so

$$Pr\{lose \text{ w} / T_2\} \geq Pr\{lose \text{ w} / T_1\}$$

$$Pr\{lose \text{ w} / T = \infty\} \geq \underbrace{Pr\{lose \text{ w} / T < \infty\}}$$

So if the gambler keeps betting until broke, he is sure to go broke.

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Return to the origin.

If you start at the origin and move left or right with equal probability, and keep moving in this way,

Pr{return to origin} = 1

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Unbiased Case for $T = \infty$

Likewise,

$E[\#\text{bets w}/T = \infty] \geq E[\#\text{bets w}/T < \infty]$
$= n(T\text{-}n) \rightarrow \infty$
(as $T \rightarrow \infty$)

So the expected #bets for the gambler to go broke is infinite!