NOTES

Do You Know Your Relative Driving Speed?

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Drivers on a busy highway or freeway typically select a driving speed based not only on the posted speed limit, but also on the velocities of nearby vehicles. Many drivers attempt to stay at or near the "flow of traffic," while others prefer to go a bit faster or slower. Traffic safety engineers have stated that the safest speed to travel on a busy freeway is at the 85th percentile of traffic speeds. A natural question arises: How can individuals gauge their speed percentiles from observing traffic in the vicinity?

In this note, I utilize a simple idealized model for traffic flow: Assume that each vehicle travels at a constant speed and that the locations of vehicles and their speeds are described by what is called a *marked Poisson process*. This means that vehicles are randomly spaced along the highway, and that their speeds are independent of their locations and of all other vehicles' speeds and locations. Assume also that the distribution of traffic speeds (the "marks") has density function $f(x)$, defined for speeds $x > 0$, continuous and positive on its support and not changing over time. That is, the number of vehicles per mile of road traveling between speeds x and $x + \Delta x$ is approximately proportional to $f(x)\,\Delta x$, when Δx is small. For example, if $f(x)$ is the uniform density on some interval $[a, b]$, then we expect an equal number of vehicles traveling at each speed between a and b.

The naïve estimate of one's percentile rank in the distribution of speeds on the highway is simply the observed proportion of vehicles passed out of the total number that one passes or is passed by. In particular, if the number of vehicles passing is equal to the number of vehicles being passed, then a driver might conclude that (s)he is driving at the median speed. Clevenson, Schilling, Watkins, and Watkins **[1]** recently showed that this is not the case. In actuality, when the number of vehicles passing equals the number being passed, the driver is, surprisingly, traveling at the *mean* speed rather than the median. More generally, the driver's speed percentile cannot be obtained merely by counting the vehicles passing and being passed.

This article explores the relationship between the naïve estimate, based on counting vehicles passing or being passed, and the actual speed percentile rank of a driver under the assumed model. We show that a driver traveling at a relatively slow or relatively fast speed will, (perhaps subconsciously) using the naïve estimate, judge his or her speed to be in a more extreme percentile than is actually the case. For instance, a person driving at the 85th percentile may perceive that (s)he is driving in an even higher speed percentile.

To begin, suppose that a particular vehicle V is traveling at speed s. Then the *actual speed percentile* of this vehicle's speed under the assumed model is $p = F(s) = \int_0^s f(x)\,dx$, while the driver's *observed speed percentile* is equal to the proportion of passed vehicles out of all other vehicles seen (either passing V or passed by V). Since vehicles at speed x will be encountered at a rate proportional to both the number of vehicles at speed x and the absolute difference between x and s, this observed percentile will converge over time to

$$p_{\text{obs}} = \frac{\int_0^s (s-x)\, f(x)\, dx}{\int_0^\infty |s-x|\, f(x)\, dx}.$$

Since F is an increasing function of s, it has an inverse, which we use to express p_{obs} as a function of p:

$$p_{\text{obs}}(p) = \frac{\int_0^{F^{-1}(p)} (F^{-1}(p)-x)\, f(x)\, dx}{\int_0^\infty |F^{-1}(p)-x|\, f(x)\, dx}. \tag{1}$$

For the simplest case, when traffic speeds are uniformly distributed, we obtain from (1) that $p_{\text{obs}}(p) = p^2/(2p^2 - 2p + 1)$. Since $p_{\text{obs}}(p)$ is not the identity function, the driver's observed percentile does not generally match his or her actual speed percentile. In fact, we find that $p_{\text{obs}}(p) < p$ for $0 < p < .5$ and $p_{\text{obs}}(p) > p$ for $.5 < p < 1$. For example, if a car is traveling in the 75th percentile of speeds, then $p_{\text{obs}}(.75) = .90$, so the driver will likely perceive that s(he) is in approximately the 90th speed percentile. A car moving in the 75th percentile of speeds will typically pass not *three* times as many vehicles as it is passed by, as one might at first expect, but *nine* times as many!

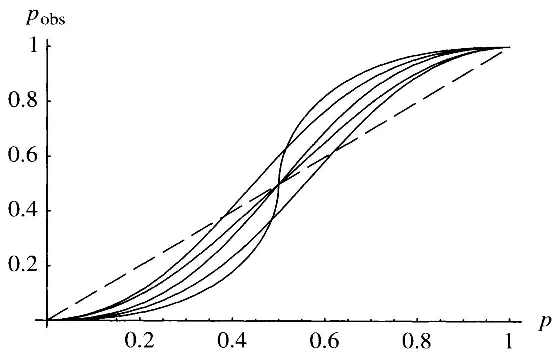

Figure 1 The observed percentile function $p_{\text{obs}}(p)$ for several traffic speed density functions

FIGURE 1 shows p_{obs} as a function of p for five different traffic speed density functions, which are shown in FIGURE 2 below. These simple models cover a wide variety of possible situations, including traffic speed distributions that are uniform, or strongly skewed towards either faster or slower speeds, or in which most drivers travel at medium speeds or at extreme speeds (either very fast or very slow). The diagonal $p_{\text{obs}} = p$ (dashed line) is shown for reference. Looking at the left half of FIGURE 1 $(0 < p < 0.5)$, the curves from left to right correspond to the traffic speed density functions $f_3(x) = 2x$, $f_4(x) = 2 - 4|x - 0.5|$, $f_1(x) = 1$ (the uniform distribution), $f_2(x) = 2(1-x)$, and $f_5(x) = 4|x - 0.5|$. The relationship between p_{obs} and p is invariant with respect to linear transformations of the traffic speed density function, so these results apply to any interval of possible speeds, for example [45 mph, 75 mph], by simply transforming the density functions in FIGURE 2 appropriately. Density functions with infinite support (such as $(0, \infty)$) show similar results, although their relevance as models for traffic speeds is dubious.

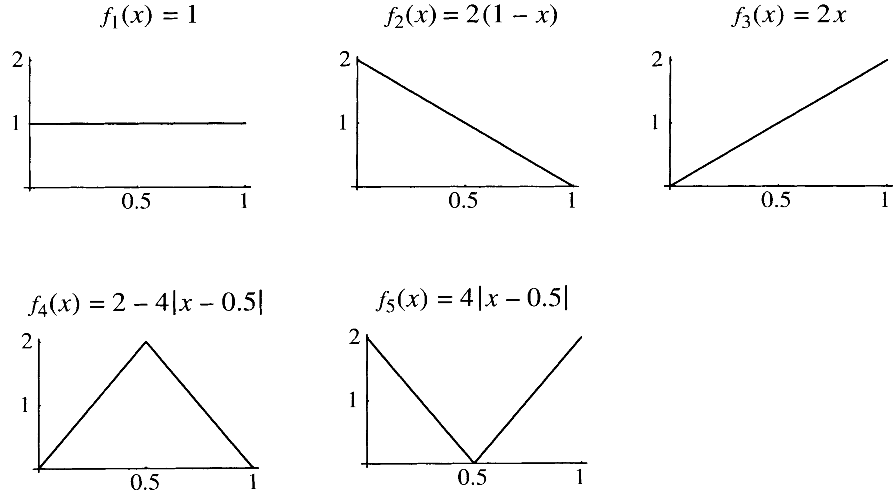

Figure 2 Traffic density functions used in FIGURE 1

We see from FIGURE 1 that in each instance there is only one speed percentile, say $\tilde{p}$, for which $p_{\text{obs}} = p$. This is nearly always the case, although exceptions do exist. The value of $\tilde{p}$ will be close to 0.5 (corresponding to the median speed of traffic) unless the traffic speed density function is highly skewed. In general, FIGURE 1 shows that a driver will tend to overestimate his or her extremity in the speed distribution, sometimes quite substantially. A driver in a high speed percentile ($> \tilde{p}$) will perceive that (s)he is in an even higher one, while a driver in a low speed percentile ($< \tilde{p}$) will think (s)he is in an even lower one.

The driver's observed speed percentile is thus a biased representation of the true speed percentile p of V unless $p = \tilde{p}$. This bias is due to the overweighting of speeds very different from s as compared to those close to s. That is, the number of vehicles a driver will see whose speeds are very different from his or her own speed overrepresents the actual number of vehicles traveling at those speeds when compared to the number of vehicles seen that are traveling at speeds similar to the driver's. For example, the number of much slower vehicles that a relatively fast driver passes is out of proportion to the actual number of such vehicles, making the driver perceive that (s)he is in an even higher speed percentile than (s)he really is.

We now show that for the stated model for traffic speeds, $p_{\text{obs}}(p)$ always takes the form shown in FIGURE 1:

THEOREM. *For any continuous density function $f(x)$ positive on its support, $p_{\text{obs}}(p)$ is a strictly increasing function, and for some p^*, $p^{**} \in (0, 1)$, we have $p_{\text{obs}}(p) < p$ for $0 < p < p^*$ and $p_{\text{obs}}(p) > p$ for $p^{**} < p < 1$.*

Proof. Write the inverse of F as $h(p) = F^{-1}(p)$. Making the substitution $u = F(x)$ in (1) yields

$$p_{\text{obs}}(p) = \frac{\int_0^p (h(p) - h(u))\,du}{\int_0^1 |h(p) - h(u)|\,du} = \frac{\int_0^p (h(p) - h(u))\,du}{\int_0^p (h(p) - h(u))\,du + \int_p^1 (h(u) - h(p))\,du}.$$

Since $h(p)$ is an increasing function, it follows at once that $\int_0^p (h(p) - h(u))\,du$ is increasing and $\int_p^1 (h(u) - h(p))\,du$ is decreasing. A little work with inequalities shows that $p_{\text{obs}}(p)$ is increasing as claimed.

That $p_{\text{obs}}(p) < p$ for $0 < p < p^*$ for some $p^* \in (0, 1)$ follows from the fact that

$$\lim_{p\to 0^+} \frac{p_{\text{obs}}(p)}{p} = \lim_{p\to 0^+} \frac{\frac{1}{p}\int_0^p (h(p) - h(u))\,du}{\int_0^1 |h(p) - h(u)|\,du} = \frac{h(0) - h(0)}{\int_0^1 |h(0) - h(u)|\,du} = 0.$$

Similarly, to show that $p_{\text{obs}}(p) > p$ for $p^{**} < p < 1$ for some $p^{**} \in (0, 1)$, we use

$$\lim_{p\to 1^-} \frac{1 - p_{\text{obs}}(p)}{1 - p} = \lim_{p\to 1^-} \frac{\frac{1}{1-p}\int_p^1 (h(p) - h(u))\,du}{\int_0^1 |h(p) - h(u)|\,du} = \frac{h(1) - h(1)}{\int_0^1 |h(1) - h(u)|\,du} = 0. \quad \blacksquare$$

Note that we can take $p^* = p^{**}$ for each density function given in FIGURE 2. Although density functions for which p^* must be less than p^{**} exist, they have unusual forms that are unlikely to represent plausible traffic speed scenarios. FIGURE 3 shows an example:

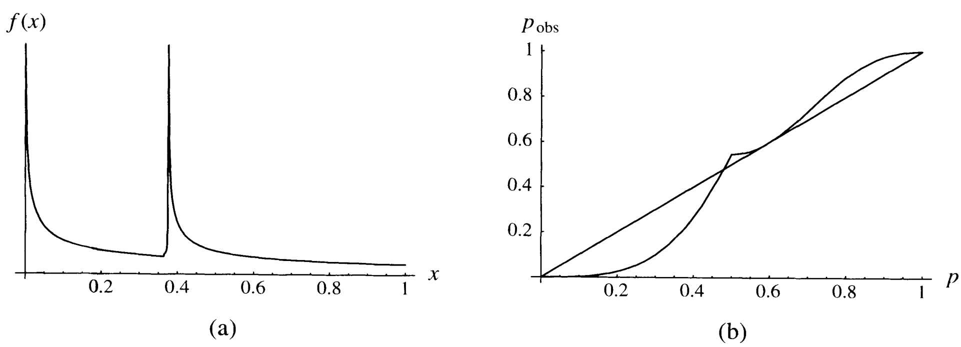

Figure 3 (a) A density function $f(x)$ with $p^* < p^{**}$; (b) $p_{\text{obs}}(p)$ for f shown in Fig. 3a ($p^* = .48$, $p^{**} = .61$)

Conclusion

Our theorem describes in simple terms the perception bias that may occur when a driver estimates the speed percentile p at which (s)he is driving by counting the number of vehicles passing or being passed. At low speeds relative to traffic, one will underestimate p, while at high speeds, one will overestimate p, regardless of the specific distribution of traffic speeds. The perception bias is greatest in situations when there is a large variation in traffic speeds (for instance, $f_5(x)$), and least when there is small variation (for instance, $f_4(x)$).

Could it be that this perception bias encourages faster drivers to slow down, and slower drivers to speed up? If so, this is not the only way in which a driver's misperceptions may affect the way he or she drives. Redelmeier and Tibrishani **[2]** showed that a driver in congested traffic may mistakenly judge that an adjacent lane is faster, perhaps leading to a needless lane change, when in fact the average speed of vehicles in that lane is just the same as in the driver's own. The phenomenon occurs because when the speeds of the two lanes fluctuate, with the same average speed for each, more of the driver's time is spent being passed by vehicles in the next lane than is spent passing such vehicles. Evidently, when it comes to judging one's speed relative to traffic, things are not generally what they seem!

REFERENCES

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Territorial Dynamics: Persistence in Territorial Species

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The widely studied and very controversial northern spotted owl, along with many other threatened and endangered species, exhibits territorial behavior. That is, adult pairs claim and defend a home range encompassing sufficient resources and of sufficient size to allow the pair to survive and reproduce successfully. Readers may be familiar with population models such as the logistic growth model, the Gompertz model, the Ricker model, and the Beverton-Holt model. These all capture the basic concept of limited growth (carrying capacity); however, they fail to exhibit some fundamental characteristics of the dynamics of territorial species. In particular, they do not exhibit a threshold in the density of suitable habitat below which the species is destined for extinction even if some suitable habitat is still available.

In this paper, we develop a model first proposed by Lamberson and Carroll [**1**] for the dynamics of a territorial animal or bird population. It consists of a continuous model for dispersal which distinguishes between *adults*—individuals who hold territories—and *juveniles*—those (nonterritorial) individuals that have not yet secured a home range. Here we think of birth not as the time of physical birth, but the time at which juveniles leave their natal territory and begin the search for their own home range. The model explicitly considers the cost of dispersal by including an ongoing rate of mortality due to predation and starvation while animals search for a territory. We establish that there is a threshold for density of suitable habitat, below which the population must decrease to extinction and above which the population tends to a stable positive equilibrium size.

Within populations of territorial animals we frequently find individuals that have not had the good fortune to secure a home range. These individuals, sometimes called floaters, usually occupy habitat of marginal quality and not suitable for attracting a mate. Usually, they eke out a secretive existence on the fringes of territories already claimed by other individuals. The dynamics of this floater population is important in understanding the overall behavior of the species, especially if the species is threatened or endangered. In our model, the floaters are considered part of the population of juveniles since they have not yet secured a suitable territory.

In this paper, we use a simple system of differential equations to describe the dynamics of a territorial species. The behavior of the system will be studied under both equilibrium and nonequilibrium conditions. For equilibrium conditions, we will establish: the fixed points, their stability, and the critical threshold in habitat density for persistence of the population.