Do Good Hands Attract?

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for $x=-1,2$). Similar methods yield reducible fourth degree polynomials (of both types).

We conclude by mentioning several papers on closely related topics. Ore [5] has generalized the above and other results for polynomials assuming a particular prime value a certain number of times to those assuming any prime values, and has also determined the exact number of prime values a reducible polynomial can assume. Weisner [7] has obtained results for polynomials of degree n which assume the same value k (where k is any integer $\neq 0$) for n distinct values of x. And finally the writer has shown [2] that finding solutions for the equal degree decomposition in Pólya's theorem is essentially equivalent to finding *ideal* solutions in the Tarry-Escott problem [4].

	-1	0	1	2	a_5	a_6
$F(x)$	1	-1	-1	1	p	p
$G(x)$	$-p$	p	p	$-p$	-1	-1
$H(x)$	$-p$	$-p$	$-p$	$-p$	$-p$	$-p$

TABLE 1

	0	a_2	a_3
$F(x)$	1	-1	p
$G(x)$	p	$-p$	1
$H(x)$	p	p	p

TABLE 2

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Do Good Hands Attract?

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There are two different opinions about the prevalence of good hands in a poker deal. According to one player: "Every time I get a good hand, everyone else drops out and I only win a small pot." According to another: "Poker is an exciting game because there are either no good hands or many good hands in a deal."

Do good hands attract? That is, does the existence of one good hand increase the probability that there will be other good hands? We shall investigate this question for ordinary five-card poker. The reader is invited to use similar methods for other games of chance.

To simplify our calculations we shall restrict ourselves to some of the best hands in five-card poker. In the sequel we shall let R_i, S_i, K_i, H_i, F_i, and T_i be the events that player i is dealt a royal flush, straight flush (including royal flushes), four of a kind, full house, flush (including straight and royal), and three of a kind, respectively.

For royal flushes we have

$$P(R_i)=\frac{4}{\binom{52}{5}}=1.539\times 10^{-6}$$

$$P(R_2|R_1)=\frac{3}{\binom{47}{5}}=1.956\times 10^{-6}$$

$$P(R_3|R_1\cap R_2)=\frac{2}{\binom{42}{5}}=2.351\times 10^{-6}$$

$$P(R_4|R_1\cap R_2\cap R_3)=\frac{1}{\binom{37}{5}}=2.294\times 10^{-6}.$$

Since $P(R_2|R_1)>P(R_2)$ we say that a royal flush is **attractive**. A measure of this attraction is the **coefficient of attraction** $a(R_2,R_1)=P(R_2|R_1)/P(R_2)=1.27$. Thus, a royal flush is 1.27 times more likely given the existence of another royal flush than it otherwise would have been. The next coefficient of attraction is $a(R_3,R_1\cap R_2)=P(R_3|R_1\cap R_2)/P(R_3)=1.53$. This indicates that a royal flush is 1.53 times more likely given the existence of two other royal flushes than it otherwise would have been. The remaining coefficients of attraction are $a(R_4,R_1\cap R_2\cap R_3)=1.49$, and $a(R_i,R_1\cap\cdots\cap R_{i-1})=0$ for $i\geqslant 5$. It is interesting that although one, two, or three royal flushes are attractive, two royal flushes are more attractive than either one or three royal flushes.

We next summarize the situation for four of a kind. Since $P(K_i)=(13)(48)/\binom{52}{5}=2.401\times 10^{-4}$ and $P(K_2|K_1)=(44/48)(12)(43)/\binom{47}{5}=3.084\times 10^{-4}$, we have $a(K_2,K_1)=1.28$. Similarly, $P(K_3,K_1\cap K_2)=(44/48)(39/43)(11)(38)/\binom{42}{5}=4.085\times 10^{-4}$, so $a(K_3,K_1\cap K_2)=1.70$. Letting $a_i=a(K_{i+1},K_1\cap\cdots\cap K_i)$ we have

$a_1=1.28$	$a_4=3.41$	$a_7=18.91$
$a_2=1.70$	$a_5=5.32$	$a_8=52.46$
$a_3=2.35$	$a_6=9.23$	$a_9=296.7.$

In this case the coefficients of attraction increase monotonically.

We list below a few other coefficients of attraction. It would be a good exercise for a class to verify these and to compute others.

$$a(F_2,F_1)=1.29\quad a(F_3,F_1\cap F_2)=1.60\quad a(F_4,F_1\cap F_2\cap F_3)=1.59$$

$$a(S_2,S_1)=1.38\quad a(H_2,H_1)=1.20.$$

Let's now consider attractions for hands of different types. For example, does a royal flush attract four of a kind? Since

$$P(K_i)=\frac{(13)(48)}{\binom{52}{5}}=2.401\times 10^{-4}$$

and

$$P(K_2|R_1)=\frac{(8)(43)}{\binom{47}{5}}=2.243\times10^{-4},$$

we see that a royal flush *repels* four of a kind. Indeed, the coefficient of attraction is $a(K_2, R_1)=P(K_2|R_1)/P(K_2)=0.93$. However,

$$P(K_3|R_1\cap R_2)=\frac{(8)(38)}{\binom{42}{5}}=3.574\times10^{-4},$$

so that two royal flushes attract four of a kind and $a(K_3, R_1\cap R_2)=1.49$. We find that three and four royal flushes also attract four of a kind. In fact, $a(K_4, R_1\cap R_2\cap R_3)=2.52$ and $a(K_5, R_1\cap R_2\cap R_3\cap R_4)=4.63$.

It turns out that a royal flush repels full houses, attracts flushes, and repels three of a kind: $a(H_2, R_1)=0.95$, $a(F_2, R_1)=1.30$, and $a(T_2, R_1)=0.97$.

We now answer the question that titles this paper: Do good hands attract? To be specific, let us define a **good hand** to be a full house or better. Of course, this is quite arbitrary. One could just as well define a good hand to be three of a kind or better, and the reader might try this or other definitions. However, the more types one admits as a good hand, the harder the calculations.

If G_i represents the event that player i is dealt a good hand, then $G_i=S_i\cup K_i\cup H_i$. Hence, $P(G_i)=P(S_i)+P(K_i)+P(H_i)=1.695\times10^{-3}$. To compute $P(G_2|G_1)$ we have

$$\begin{aligned}P(G_2|G_1)&=P(S_2\cup K_2\cup H_2|S_1\cup K_1\cup H_1)\\&=P[(S_2\cup K_2\cup H_2)\cap(S_1\cup K_1\cup H_1)]/P(G_1)\\&=P(G_1)^{-1}[P(S_1)P(S_2|S_1)+P(K_1)P(K_2|K_1)+P(H_1)P(H_2|H_1)\\&\quad+2P(S_1)P(K_2|S_1)+2P(S_1)P(H_2|S_1)+2P(K_1)P(H_2|K_1)].\end{aligned}$$

After computing all the above probabilities one finds that $P(G_2|G_1)=2.063\times10^{-3}$. Hence, one good hand attracts another, and $a(G_2,G_1)=P(G_2|G_1)/P(G_2)=1.22$. Of course, the answer could be quite different if the standard for a good hand is lowered. (One could also consider whether two good hands attract a good hand, and so forth.)

These examples motivate a theory of attraction for an arbitrary probability space. To avoid certain pathologies, we shall only consider events A such that $0<P(A)<1$. We say that an event A **attracts** an event B if $P(B|A)>P(B)$. If $P(B|A)<P(B)$ we say that A **repels** B. Since $P(B|A)=P(A\cap B)/P(A)$, we see that A attracts B if and only if $P(A\cap B)>P(A)P(B)$, and A repels B if and only if $P(A\cap B)<P(A)P(B)$. It follows that A attracts B if and only if B attracts A so attraction is a symmetric relation. Hence, we can use the terminology that A and B are **mutually attractive** instead of A attracts B. Similar terminology can be used for repulsion.

It is clear that A attracts A, so that attraction is a reflexive relation. Also, A repels A', the complement of A. More generally, if either A or B is contained in the other, then A and B are mutually attractive. Moreover, if A and B are disjoint, then A and B are mutually repulsive.

Attraction is not, however, an equivalence relation, since it is not transitive. For example, in a probability space of five points, a_1, a_2, a_3, a_4, a_5, each with equal probability, let $A=\{a_1, a_2, a_3\}$, $B=\{a_2, a_3, a_4\}$ and $C=\{a_3, a_4, a_5\}$. Then A and B are mutually attractive, as are B and C, but A and C are **not** mutually attractive. For a similar continuous example, select the unit interval $[0,1]$ as the probability space, with Lebesgue measure, and let $A=[0,\frac{1}{3}]$, $B=[\frac{1}{6},\frac{1}{2}]$, and $C=[\frac{1}{3},\frac{2}{3}]$.

We conclude with a list of good class exercises:

Problem 1. A and B are mutually attractive if and only if $P(B|A)>P(B|A')$.

Thus A and B are mutually attractive if and only if B is more likely when A has occurred than when A has not occurred.

Problem 2. A neither attracts nor repels B if and only if A and B are stochastically independent.

Problem 3. If A attracts B, then A repels B'.

Problem 4. A and B are mutually attractive if and only if A' and B' are mutually attractive.

Problem 5. If $B \cap C = \varnothing$ and A attracts both B and C, then A attracts $B \cup C$.

Problem 6. If A attracts both B and C, and A repels $B \cap C$, then A attracts $B \cup C$. Is there any example in which A attracts both B and C, but repels $B \cup C$?

Problem 7. If $B_1, \ldots, B_n$ are mutually disjoint and exhaustive ($\cup B_i = S$), and if A attracts some B_i, then A must repel some B_j.

We can define the **coefficient of attraction** for two events A and B by $a(A, B) = a(B, A) = P(A \cap B)/P(A)P(B) = P(A|B)/P(A) = P(B|A)/P(B)$. We can then use the coefficients of attraction to express Bayes' rule:

Problem 8. If $B_1, \ldots, B_n$ are mutually disjoint and exhaustive, then $\Sigma a(A, B_i)P(B_i) = 1$.

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Factorization of a Matrix Group

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Matrix theory is a good source for illustrations of the basic concepts of group theory. For example, if A is any element of the group $\mathrm{GL}(n, R)$ of nonsingular $n \times n$ matrices, and $\lambda = \det A$, then whenever $\lambda^{1/n}$ is real, we can write A as $(\lambda^{-1/n}A)(\lambda^{1/n}I)$ where I is the identity matrix. Since $\det(\lambda^{-1/n}A) = 1$, the matrix $\lambda^{-1/n}A$ is in the group $\mathrm{SL}(n, R)$ of matrices with determinant 1; moreover, $\lambda^{1/n}I$ is a scalar matrix which is in the center of $\mathrm{GL}(n, R)$. Each of these subgroups is normal in G and their intersection is the subgroup $\{I, -I\}$ when n is even, and $\{I\}$ when n is odd. Thus when n is odd, it follows that $\mathrm{GL}(n, R)$ is the direct product of these two subgroups.

This factorization of A brings to mind the theorem that $\mathrm{GL}(n, K)$ is a semidirect product (where only one of the two subgroups need be normal) of the unimodular matrices $\mathrm{SL}(n, K)$ and the nonzero field elements K^* [1; 158], and motivates the following quick proof. If K is any field and $A = [a_{ij}]$ is any matrix in $\mathrm{GL}(n, K)$ with $\det(A) = \lambda$, then

$$A = \begin{bmatrix} a_{11} \cdots & a_{1n-1} & a_{1n}\lambda^{-1} \\ a_{21} \cdots & a_{2n-1} & a_{2n}\lambda^{-1} \\ \vdots & \vdots & \vdots \\ a_{n1} \cdots & a_{nn-1} & a_{nn}\lambda^{-1} \end{bmatrix} \begin{bmatrix} 1 & 0 \ldots & & 0 \\ 0 & 1 \ldots & & 0 \\ \vdots & & \ddots & \vdots \\ & & 1 & \\ 0 & 0 \ldots & & \lambda \end{bmatrix}.$$

The left matrix in this factorization is unimodular, whereas the right matrix is in a nonnormal subgroup H of $\mathrm{GL}(n, K)$, and H is isomorphic to K^*. Thus, the factorization of A makes the result that $\mathrm{GL}(n, K)$ is a semidirect product of $\mathrm{SL}(n, K)$ and K^* apparent. It is easy to see that the intersection of the two subgroups here is trivial.

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