A Rule of Thumb (not only) for Gamblers

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Abstract

Let prize X in a game be a random variable with a cumulating distribution function F, $E[X] \neq 0$, and $\mathrm{Var}(X) < \infty$. In a Gambler's Ruin Problem we consider the probability $P_F(A, B)$ of accumulating fortune A before losing the initial fortune B. Suppose our Gambler is to choose between different strategies with the same expected values and different variances. $P_F(A, B)$ is known to depend in general on the whole cumulative distribution function F of X. In the paper we derive an approximation which implies the following rule called A Rule of Thumb (not only) for Gamblers:
if $E[X] < 0$ *then the strategy with the greater variance is superior while in case* $E[X] > 0$ *the strategy with the smaller variance is more favorable to the Gambler.*

We include some examples of applications of The Rule. Moreover we derive a general solution in the Roulette case and use it to show good behavior of The Rule explicitly.

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1. INTRODUCTION.

Let X, X_i $i = 1, \ldots, n$ be independent, identically distributed random variables (r.v.) with a cumulative distribution function (c.d.f.) F and let

$$S_n = \sum_{i=1}^{n} X_i. \tag{1.1}$$

A modification of the classical Gambler's Ruin Problem can be formulated in the following way in terms of the random walk theory. Let X_i's represent prizes (or losses if $X_i < 0$) of the Gambler in consecutive plays of the game. Assuming that the Gambler begins with an initial fortune $B > 0$, he withdraws at the first time N either after losing his initial fortune B (Gambler's ruin) or after successfully accumulating fortune $A > 0$. It is known (see Ross (1983), p.234–235) that probability $P_F(A, B)$ of the success equals the probability that random walk S_N reaches level A or above before reaching a value less or equal $-B$.

For the classical Gambler's Ruin Problem we refer to Whitworth (1901), Uspensky (1937) and to more recent texts by Dubins and Savage (1965), Feller (1966), pp. 344-349, Ross (1983), pp. 235, Billingsley (1979), p. 77 to list a few references. Another, so called *attrition ruin problem* was considereded in Kaigh (1979).

The formulation of the problem in terms of the Gambler's ruin is simple and attractive, and the scheme provides simple and reasonable models for various types of activities and phenomena (cf. Examples 1-3 below).

Let $\theta_F \neq 0$ be the unique non-zero root of the equation

$$\phi(\theta) = E[e^{\theta \cdot X}] = 1. \tag{1.2}$$

It is easy to see that such θ_F exists when the moment generating function is finite and

$$P(X > 0) > 0 \text{ and } P(X < 0) > 0.$$

It will be convenient to consider a modified version of equation (1.2) by taking logarithms of both sides

$$\psi(\theta) = \ln(E[e^{\theta \cdot X}]) = 0. \tag{1.3}$$

Jensen's inequality and strict concavity of ψ imply that

$$\theta_F \cdot E[X] < 0. \tag{1.4}$$

Using the martingale approach one can derive the following formulas (see Ross (1983), p. 235) for the probability $P_F(A, B)$ of a successful termination of the game

$$P_F(A, B) = \frac{1 - e^{-\theta_F \cdot B}}{e^{\theta_F \cdot A} - e^{-\theta_F \cdot B}} + \epsilon_F(A, B), \tag{1.5}$$

where

$$\begin{aligned}
\epsilon_F(A,B) &= \frac{w_F(A,B)}{e^{\theta_F \cdot A} - e^{-\theta_F \cdot B}}, && (1.6)\\
w_F(A,B) &= -u_F(A,B)P_F(A,B) - v_F(A,B)(1-P_F(A,B)), && (1.7)\\
u_F(A) &= E_F(\exp[\theta_F S_N] | S_N \geq A) - \exp(\theta_F A), && (1.8)\\
\text{and} & \\
v_F(B) &= E_F(\exp[\theta_F S_N] | S_N \leq -B) - \exp(-\theta_F B). && (1.9)\\
& && (1.10)
\end{aligned}$$

Notice that

$$u_F(A) \cdot v_F(B) \leq 0. \tag{1.11}$$

Hence

$$|\epsilon_F(A,B)| \leq \frac{\max(|u_F(A)|, |v_F(B)|)}{|e^{\theta_F \cdot A} - e^{-\theta_F \cdot B}|} \tag{1.12}$$

and

$$\begin{aligned}
u_F(A) &= 0 \Longrightarrow \epsilon_F(A,B) \geq 0, && (1.13)\\
v_F(B) &= 0 \Longrightarrow \epsilon_F(A,B) \leq 0. && (1.14)
\end{aligned}$$

Often error $\epsilon_F(A,B)$ is negligible and (1.5) yields a useful approximation to $P_F(A,B)$ given by

$$\bar{P}_F(A,B) = \frac{1 - e^{-\theta_F \cdot B}}{e^{\theta_F \cdot A} - e^{-\theta_F \cdot B}}. \tag{1.15}$$

Let us note that Uspenski (1937), p.146 obtained — for random variables X taking only two values — estimates of $P_F(A,B)$ from above and below. In the sequel we shall use (1.5) to keep control over the error term $\epsilon_F(A,B)$. Approximation (1.15) implies the following ones for the expected value of the game $E[S_N]$

$$E[S_N] \approx A \cdot P_F(A,B) - B \cdot (1 - P_F(A,B)) \tag{1.16}$$

and for the expected time of the duration of the game $E[N]$

$$E[N] \approx \frac{A \cdot P_F(A,B) - B \cdot (1 - P_F(A,B))}{E[X]}. \tag{1.17}$$

In view of the Wald equality

$$E[S_N] = E[N] \cdot E[X] \tag{1.18}$$

and hence the expectation $E[X]$ is critical for the quality of a strategy.

In this note we derive a quick and easily to applied approximation to the exact test for comparison of performances of gambling strategies corresponding to X's with equal expectation and different variances. Since the conclusions from the approximate criterion are correct in typical cases we call it a *Rule of Thumb (not only) for Gamblers.* The Rule can be formulated as follows

$$\textit{if } E[X] < 0 \textit{ then the strategy with the greater variance is superior while in case } E[X] > 0 \textit{ the strategy with the smaller variance is more favorable to the Gambler.} \tag{1.19}$$

The Rule may serve as a useful tip both in classroom and in numerous models just as a *rule of thumb* in the process of fast decision making. Below we include several typical examples of applications of The Rule. A precise comparison of the exact solution (1.5) and approximation (1.15) in the Roulette case is postponed till Section 3. The comparison provides also examples in which — because of non-negligible error terms — The Rule is not valid and results in an inverse ordering of strategies.

Example 1. *At a water reservoir daily changes of the water level can be described by a random variable X. Let us measure the water levels as deviation from a 'standard' state at level 0. The accumulation of these random changes may result in crossing either the lower critical level $-B$ or an upper critical level A. Suppose that in a considered period of time the mean daily change is negative. In a year (or in a climate) with large rainfall fluctuation the chances for crossing the upper critical level A are (according to The Rule and in agreement with observations) much higher than in a year (or climate) with a stable weather and having the same average daily rainfall.*

Example 2. *An investor on a Stock Market is to choose between two types of stocks: one is characterized by small fluctuation while the other is very speculative. Assume that an average return from investing on both types of stocks is the same. Buying and eventually selling the stock results in a random gain or loss. Subsequent operations result in accumulation of incomes and it can be modeled as a game described at the begining of this section. The Rule implies that on a 'bear market' (i.e. when the average of stock prices is going down) the chances of achieving fortune A before losing the initial capital B are better for the investor in speculative stocks. On a 'bull market' (i.e. when average of stock prices is gaining on value) dealing with stocks having a stable upward trend results in a higher probability of the success.*

Example 3. *Mutations in the genetic code result in average in a regression of the characteristics of the mutant. This seems intuitively clear because only some very specific mutations raise the survival skills of the species to a higher level. The average effect of chemically admissible but random mutations seems to handicap the species. It agrees with the Second Principle of Thermodynamic which requires an increase of Entropy i.e. non-equilibrium processes are moving towards the most probable state of the system. Let X and Y stand for two competing quantitative descriptions of the changes resulting from random mutations. Let these two types of mutations have the same negative expectation and let $\mathrm{Var}(X) < \mathrm{Var}(Y)$. Assume that accumulation of changes to level A results in new species on the higher level of the evolution tree. Assume further that the species die if the changes accumulate in a wrong* *direction to level $-B$. The rule implies that chances for a qualitatively positive change resulting from accumulation of subsequent genetic changes are higher in the case of mutation type Y with larger variance. It corresponds to mutation type resulting in frequent small negative changes and rare but significant mutations in the positive direction which agrees with the experience of biologists. They have observed that chains in the evolutionary processes are often missing and transitions from one species to another are often not continuos. This seems to correspond to the pace of the evolutionary process resulting from the mutation processes Y with large variances. We should however remember that the probability of achieving the high level A in a process with negative drift is very small. This in turn agrees with the sparseness (uniqueness ?) of the life in our known Universe.*

The models presented in these examples is very simplified but points to important differences in some competitive processes having the same drift but different variances.

2. THE RULE OF THUMB.

The probability of success $P_F(A,B)$ given by (1.5) is important for any characterization of the quality of the strategy X. Both approximations to the expected time of the game $E[N]$ and the expected award to the gambler $E\left[S_N\right]$ are linear functions of $P_F(A,B)$. Hence, in this paper, we focus on $P_F(A,B)$.

We recall that formulas (1.15)–(1.17) are approximate because of the overshooting effect. This occurs when the game terminates either after a win resulting in an increase of the Gambler's fortune from a value below A to a value greater than A, or when a loss results in a drop in the Gambler's fortune from a value above $-B$ to a value smaller than $-B$. The approximation is a consequence of the assumption that the final fortune of the Gambler equals A in the first case and $-B$ in the second. When A and B are large compared with values of X the approximation is satisfactory and is usually accepted in the literature (see Ross (1983), p. 235).

In the sequel we shall consider yet another approximation. Our assumptions imply that ψ given by (1.3) is a convex function with a root $\theta_F \neq 0$. The Taylor expansion of ψ at zero yields

$$\psi(\theta) = \theta \cdot E[X] + \frac{\theta^2}{2} \cdot \mathrm{Var}(X) + o(\theta^2). \tag{2.1}$$

Neglecting the 'little o' term in (2.1) we get the following approximation $\bar{\theta}_F$ to the non-zero root θ_F of equation (1.3)

$$\bar{\theta}_F = -\frac{2 \cdot E[X]}{\mathrm{Var}(X)}. \tag{2.2}$$

If expansion (2.1) of ψ is accurate at θ_F then approximation (2.2) is fairly good. We should point out that if X is normally distributed then ψ is a quadratic function and the $o(\theta^2)$ term in (2.1) equals 0. Thus *our approximation of θ_F is exact in case of normal X*. We shall need the following lemmas.

Lemma 1. *For any positive A and B function γ given by*

$$\gamma(x) = \frac{1 - e^{-Bx}}{e^{Ax} - e^{-Bx}} \tag{2.3}$$

is decreasing.

Proof: Since $\exp(x)$ is convex and increasing the The Mean Value Theorem implies that for every x

$$\frac{1 - e^{-Ax}}{A} < \frac{e^{Bx} - 1}{B}$$

Next we note that

$$\lim_{h \to 0} \frac{e^{(A+B)h} - e^{Bh}}{e^{Bh} - 1} = \frac{A}{B}$$

and hence for h sufficiently small inequality

$$(1 - e^{-Ax}) < (e^{Bx} - 1) \cdot \frac{e^{(A+B)h} - e^{Bh}}{e^{Bh} - 1}$$

holds. With some elementary algebra this can be transformed for $h > 0$ into inequality

$$\frac{1 - e^{-B(x+h)}}{e^{A(x+h)} - e^{-B(x+h)}} < \frac{1 - e^{-Bx}}{e^{Ax} - e^{-Bx}}$$

which implies that γ is decreasing. $\diamondsuit$

Lemma 2. *Let F_1 and F_2 be cumulative distribution functions, θ_i $i = 1, 2$ be the corresponding roots of (1.2), and $\bar{\theta}_i$ be approximations given by (2.2). Then inequality*

$$\begin{aligned} \gamma(\bar{\theta}_2) - \gamma(\bar{\theta}_1) \quad \geq \quad & \Big(\gamma(\theta_1) - \gamma(\bar{\theta}_1)\Big) + \epsilon_{F_1}(A, B) \\ & - \Big(\gamma(\theta_2) - \gamma(\bar{\theta}_2)\Big) - \epsilon_{F_2}(A, B) \end{aligned} \tag{2.4}$$

holds if and only if

$$P_{F_2}(A, B) \geq P_{F_1}(A, B). \tag{2.5}$$

Proof: The proof of equivalence of inequalities (2.4) and (2.5) follows easily from (1.5) and from

$$P_{F_i}(A, B) = \gamma(\bar{\theta}_i) + \Big(\gamma(\theta_i) - \gamma(\bar{\theta}_i)\Big) + \epsilon_{F_i}(A, B), \quad i = 1, 2.$$

$\diamondsuit$

Theorem 1. *Let X and Y represent random rewards of the Gambler such that*

$$\mu = E[X] = E[Y] \neq 0 \text{ and } \mathrm{Var}(X) < \mathrm{Var}(Y).$$

Moreover, let F and G be the cumulative distribution functions of X and Y, respectively such that the corresponding non-zero roots of equation (1.3) are unique. Then

(a) *if $\mu < 0$, the overshooting error is negligible, and the Taylor expansion (2.1) is sufficiently accurate in case of X and Y, i.e. if inequality*

$$\gamma\left(-2\frac{\mu}{Var(Y)}\right) - \gamma\left(-2\frac{\mu}{Var(X)}\right) \geq \left(\gamma(\theta_F) - \gamma\left(-2\frac{\mu}{Var(X)}\right)\right) + \epsilon_F(A,B) - \left(\gamma(\theta_G) - \gamma\left(-2\frac{\mu}{Var(Y)}\right)\right) - \epsilon_G(A,B) \tag{2.6}$$

holds, then it follows that $P_F(A,B) \leq P_G(A,B)$;

(b) *if $\mu > 0$, the overshooting error is negligible, and the Taylor expansion (2.1) is sufficiently accurate in case of both X and Y, i.e. if inequality*

$$\gamma\left(-2\frac{\mu}{Var(X)}\right) - \gamma\left(-2\frac{\mu}{Var(Y)}\right) \geq \left(\gamma(\theta_G) - \gamma\left(-2\frac{\mu}{Var(Y)}\right)\right) + \epsilon_G(A,B) - \left(\gamma(\theta_F) - \gamma\left(-2\frac{\mu}{Var(X)}\right)\right) - \epsilon_F(A,B) \tag{2.7}$$

holds then it follows that $P_F(A,B) \geq P_G(A,B)$.

Proof: Theorem 1 follows easily from Lemma 2 applied in the case of F and G, respectively. $\diamondsuit$

We conclude this section with several remarks.

Remark 1. *In view of Lemma 1, the left hand sides of inequalities (2.6) and (2.7) are positive while the right hand sides are typically small so that both inequalities often hold. Recall that in the case of a normal X and Y the expressions on the right hand sides of (2.6) and (2.7) equal zero and if*

$$u_F(A) = 0 \ \textit{and} \ \ u_G(A) = 0$$

or

$$v_F(B) = 0 \ \textit{and} \ \ v_G(B) = 0$$

then the errors $\epsilon_F(A,B)$ and $\epsilon_G(A,B)$ are of the same signs.

By Lemma 1 if inequalities (2.6) and (2.7) hold, Theorem 1 can be given the form referred to as The Rule of Thumb (not only) for Gamblers and formulated in Section 1 (1.19). The interpretation of The Rule is fairly easy:

if $\mu < 0$ then the Gambler has a very small chance of a sequence with sufficiently frequent events which are favorable to him thus successfully rising his fortune above A. Therefore it pays to seek rare large rewards while paying small penalties for frequent losses;

if $\mu > 0$ then the situation is opposite and playing patiently a strategy with small variance and positive expected value is recommended — time and patience have better chances of paying off.

Remark 2. *If values A and B are large compared with values of possible rewards to the player in a single game, then one can replace the original random variable X with a sum of its several independent subsequent realizations. Since the sum can be considered as approximately normally distributed, and since the Taylor approximation (2.1) is exact in the normal case, one can expect that approximation (2.2) is fairly good in typical applications.*

Remark 3. *The Taylor expansion (2.1) is taken at zero and it is important for it to yield an accurate approximation of ψ at θ_F. Formula (2.2) indicates that θ_F is typically in the vicinity of zero when $E[X] \approx 0$. Thus the chance for a good approximation increases in games with reward close to zero. The stochastic model corresponding to the described gambling seems to be adequate for numerous biological and economic mechanisms: hence the words 'not only' in the title. Since nonequilibrium processes occurring in the real world are not far from their equilibrium state, one can expect that the corresponding expectation of X is typically close to zero, and hence that our simple Rule of Thumb is correct in ordering different 'gambling' strategies.*

Remark 4. *The steps of the proof of the Theorem reveal the following conclusions about changing stakes. Changing stakes corresponds to multiplying the random variable X by a constant $s > 0$. Thus Wald's equation (1.2) implies that the new root θ_{F_s} follows the rule*

$$\theta_{F_s} = \frac{1}{s} \cdot \theta_F$$

Since function $\gamma(x)$ is decreasing, multiplying stakes by s results in an increase of the approximation to the probability of success $\bar{P}_F(A, B)$ given by (1.15) when $\mu < 0$ and $s > 1$ or when $\mu > 0$ and $s < 1$. It is clear that $\bar{P}_F(A, B)$ decreases in both cases.

3. ROULETTE

In this section we consider in detail different betting strategies in a Roulette game. We derive a general exact solution for the probability of a successful accumulation of capital $A \geq 0$ before losing the initial capital $B \geq 0$. We use this solution to show the range of parameters A and B for which The Rule of Thumb applies. [1] We consider strategies $X_k,\ k = 1,2,3,4,6,9,12,18$ in the Roulette with

$$X_k = \frac{36}{k} - 1 \text{ with probability } \tfrac{k}{37}, \tag{3.1}$$

$$X_k = -1 \text{ with probability } 1 - \tfrac{k}{37}. \tag{3.2}$$

In every game the player applying strategy X_k is choosing k different numbers from the set $\{0, \cdots, 36\}$ and puting a unit value chip on these numbers. He either loses his chip with probability $\frac{k}{37}$ when the winning number is different from any of his chosen k numbers or, in case the winning number equals one of his chosen numbers he is being awarded chips of $\frac{36}{k} - 1$ units value. The expectation for all strategies equals $\mu = E[X_k] = -\frac{1}{37}$ while the variance depends on k

$$\mathrm{Var}(X_k) = \left(\frac{36}{37}\right)^2 \left(\frac{37}{k} - 1\right). \tag{3.3}$$

To list a few examples of classic strategies covered by our scheme we note that choices $k = 1,2,3,4,6,12,$ and 18 correspond to popular strategies allowed in casinos and called *Strait, Split, Street, Square, Line, Column,* and *Black Diamond*, respectively. Playing on Black Diamond (or on Red Diamond, Pair, Impair, Passe, Manque, etc.) corresponds in our convention to strategy X_{18}. Let us note that in practice for any k, strategy X_k can be played by putting k chips on k different numbers and just considering the value of the k chips being a unit. [2] To make the comparison of different strategies easier we restrict ourselves to k's with integer win $w_k = \frac{36}{k} - 1$. The player can then put — in case of the X_k strategy — one chip on k chosen numbers.

We could not find in the literature the general form of $P_F(A,B)$ for strategies X_k and only the easiest case $k = 18$ has been solved in the quoted in Introduction standard text-books. Dubins and Savage (1956), Ch. 6, discuss the problem in general terms but get no explicit solution. Uspensky (1937) derived upper and lower bounds for the corresponding probabilities. Since the general solution is essentially used in our discussion we include its short derivation. It also provides a pattern for solutions in the case of other similar problems. It may be of some interest that the strategies of playing on k different numbers have more sophisticated properties in case of a relatively small A, see Figure 4 for one particular case. A detailed discussion of practical strategies in Roulette is however beyond the scope of this note. First we prove the following theorem which easily implies the solution in the case discussed.

[1] Casinos over the world may have rules differing slightly from these considered in Section 3. The differences may result in conclusions different from these obtained in the paper. Moreover, our Rule of Thumb is not valid in the considered Roulette case for small A or B, cf. Figures 2-4.

[2] For $k = 1,2,3,4,6,9,12,$ and 18 the awards X_k are integer and results of Theorem 2 apply. For other integer k's Corollary 1 presents a good approximation to the exact solution, which is a bit more complicated and is not discussed in this section.

Theorem 2. *Let w, B, C be integers, where*

> *B is the initial fortune of the Gambler, $B \geq 0$,*
> *C is the ultimate fortune of the Gambler, $C \geq B \geq 0$,*
> *w is the number of chips the Gambler may win in every game, $w \geq 1$.*

Suppose that in every game the Gambler is winning w chips with probability p, or losing one chip with probability $q = 1 - p$. If the game terminates either at the first moment when (a) the total fortune of the Gambler reaches or exceeds C, or (b) his fortune reaches value 0, then the probability of the former, favorable for the Gambler case is given by (3.4)-(3.6), (3.9)-(3.10), and (3.13)-(3.15).

Proof: Let w and C be fixed and consider the probability

$$\mathcal{P}(B) = \mathcal{P}(w, C, B)$$

of concluding the game for the Gambler with at least C tokens. We have the following boundary conditions

$$\mathcal{P}(B) = 0 \text{ for } B \leq 0, \text{ and} \tag{3.4}$$
$$\mathcal{P}(B) = 1 \text{ for } B \geq C. \tag{3.5}$$

If $C \leq w + 1$ then

$$\mathcal{P}(B) = 1 - q^B \tag{3.6}$$

for $B = 0, \ldots, C - 1$.

If $C > w + 1$ then the standard argument (see Dubins and Savage (1965), Ch. 6 or Billingsley (1979, p.77)) implies the following recurrence relation for $1 \leq B \leq C - w$

$$\mathcal{P}(B) = p \cdot \mathcal{P}(B + w) + q \cdot \mathcal{P}(B - 1). \tag{3.7}$$

If $B > C - w$ then

$$1 - \mathcal{P}(B) = q \cdot (1 - \mathcal{P}(B - 1)). \tag{3.8}$$

We seek the solution of (3.7)–(3.5) for $1 \leq B \leq C - w$ in the form

$$\mathcal{P}(B) = \alpha \cdot \rho^B + \beta, \tag{3.9}$$

where ρ is the unique different from 1 solution of the equation

$$p \cdot \rho^w + q \cdot \rho^{-1} = 1. \tag{3.10}$$

Function (3.9) is a solution of the linear difference equation (3.7) increasing in B, cf. Gelfond (1958), p. 272 (cf. also Uspenskii (1937), p.146). By (3.4)-(3.5) equation (3.7) yields the following two equations for $B = 1$ and $B = C - w$, respectively

$$\alpha \cdot \rho + \beta = p(\alpha \cdot \rho^{w+1} + \beta) \tag{3.11}$$
$$\alpha \cdot \rho^{C-w} + \beta = p + q(\alpha \cdot \rho^{C-w-1} + \beta). \tag{3.12}$$

Solving (3.11)-(3.12) we get

$$\alpha = \frac{1}{\rho^C - 1} \tag{3.13}$$

$$\beta = -\frac{1}{\rho^C - 1}. \tag{3.14}$$

By (3.8), (3.9), (3.13), and (3.14) we obtain the remaining part of the solution

$$\mathcal{P}(C - w + j) = 1 - q^{j+1} \cdot \frac{\rho - 1}{\rho - q} \cdot \frac{\rho^C}{\rho^C - 1}, \qquad j = 1, \ldots, w - 1 \tag{3.15}$$

Thus, the complete solution is given by (3.4)-(3.6), (3.9)-(3.10), and (3.13)-(3.15). $\diamondsuit$

Corollary 1. *The probability of a successful termination of the game in the Roulette case and using strategy X_k is given by (3.4)-(3.6), (3.9)-(3.10), and (3.13)-(3.15), where*

$$C = A + B$$

$$p = \frac{k}{37}, \text{ and } w = \frac{36}{k} - 1.$$

The expectation μ of X_k in Roulette is negative and The Rule of Thumb (not only) for Gamblers implies that higher probabilities of achieving fortune A before running off the initial fortune B are expected for strategies X_k with larger variance, i.e. with smaller k. Hence the best strategy is expected for $k = 1$.

Figure 1 shows the graphs of the approximations of the probability of success $\bar{P}_F(A, B)$ given by (1.15). The plotted functions have argument A while value of B is fixed and has a moderate value $B = 40$. The graphs for other values of B show similar behavior and are in a good agreement with The Rule. It worth noting the big difference between strategies for $k = 1$ and for $k = 18$ corresponding to strategies called Straight and Black Diamond (or Red Diamond, Pair, etc.), respectively. The Black Diamond or Red Diamond strategy is, under the presented rules and for moderate or large A, one of the worst of those considered. The strategies with $k = 18$ correspond to the classical Gambler's Ruin Problem are already classical in the literature. They have been considered e.g. in Dubins and Savage (1965), Feller (1966) pp. 344–349, Rényi (1969), Chow, Robbins and Siegmund (1971), Billingsley (1979) p. 77, and Ross (1983) p. 235.

Figure 2 displays the range of A and B for which The Rule correctly classifies X_1 strategy as the best. Figure 3 provides some information on the precision of the discussed approximations. It is worth noticing that in the present example $\bar{P}_F(A, B)$ given by (1.15) overestimates the true probability $P_F(A, B)$ of the successful termination of the game. This can be easily explained by (1.14) because the overshooting error is $v_F(B) = 0$ in the case discussed. Finally, Figure 4 shows the exact probabilities $P_F(A, B)$ corresponding to strategies called Strait ($k = 1$), Square ($k = 4$), and Black Diamond ($k = 18$), respectively. Except for small

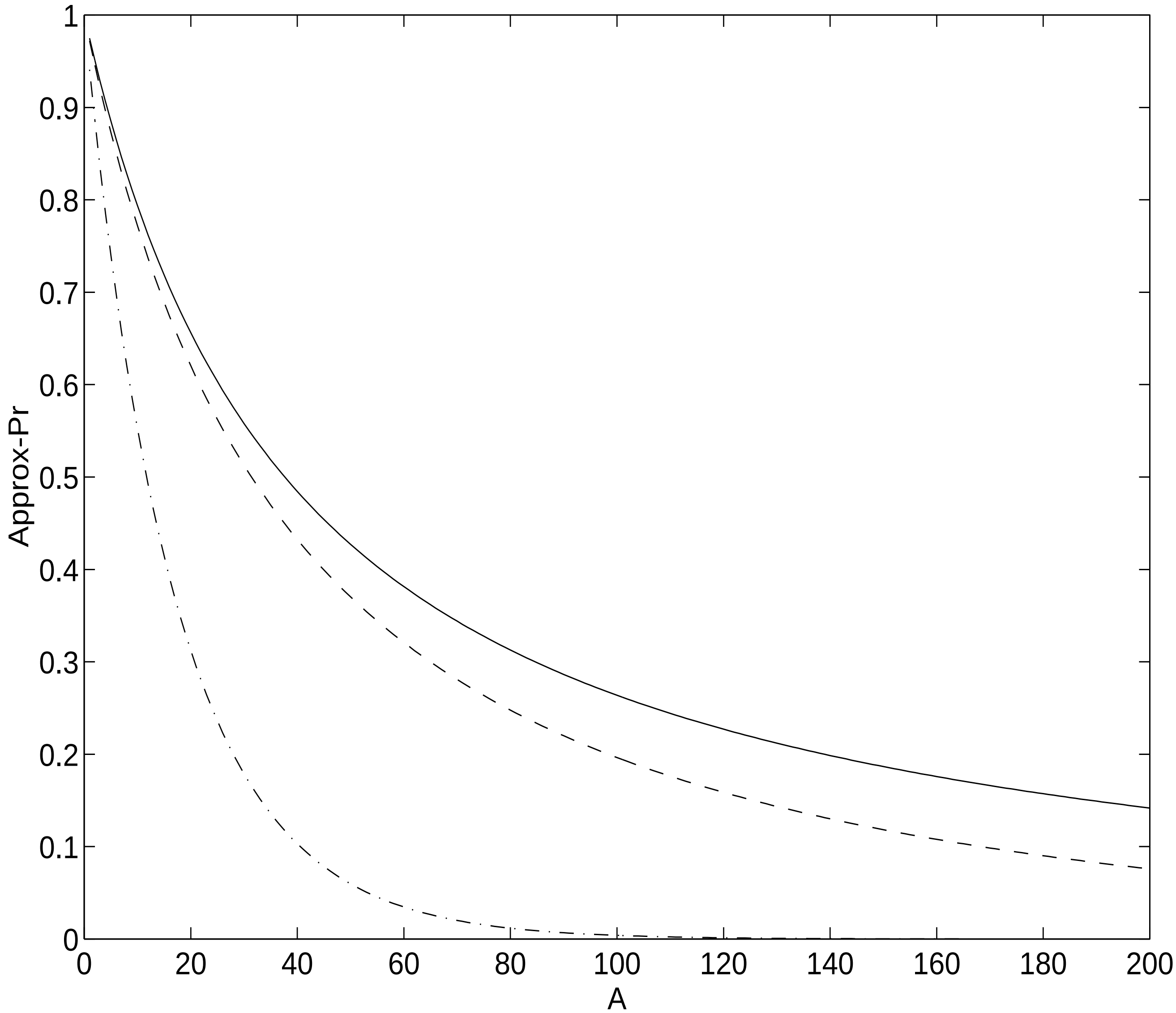

Figure 1: *Approximate probabilities* $\bar{P}_F(A, B)$ *given by (1.15) of achieving* A *before losing* B *in Roulette, for* $B = 40$ *and strategies corresponding to* $k = 1$ *(solid line),* 4 *(dashed line), and* 18 *(dashdot line).*

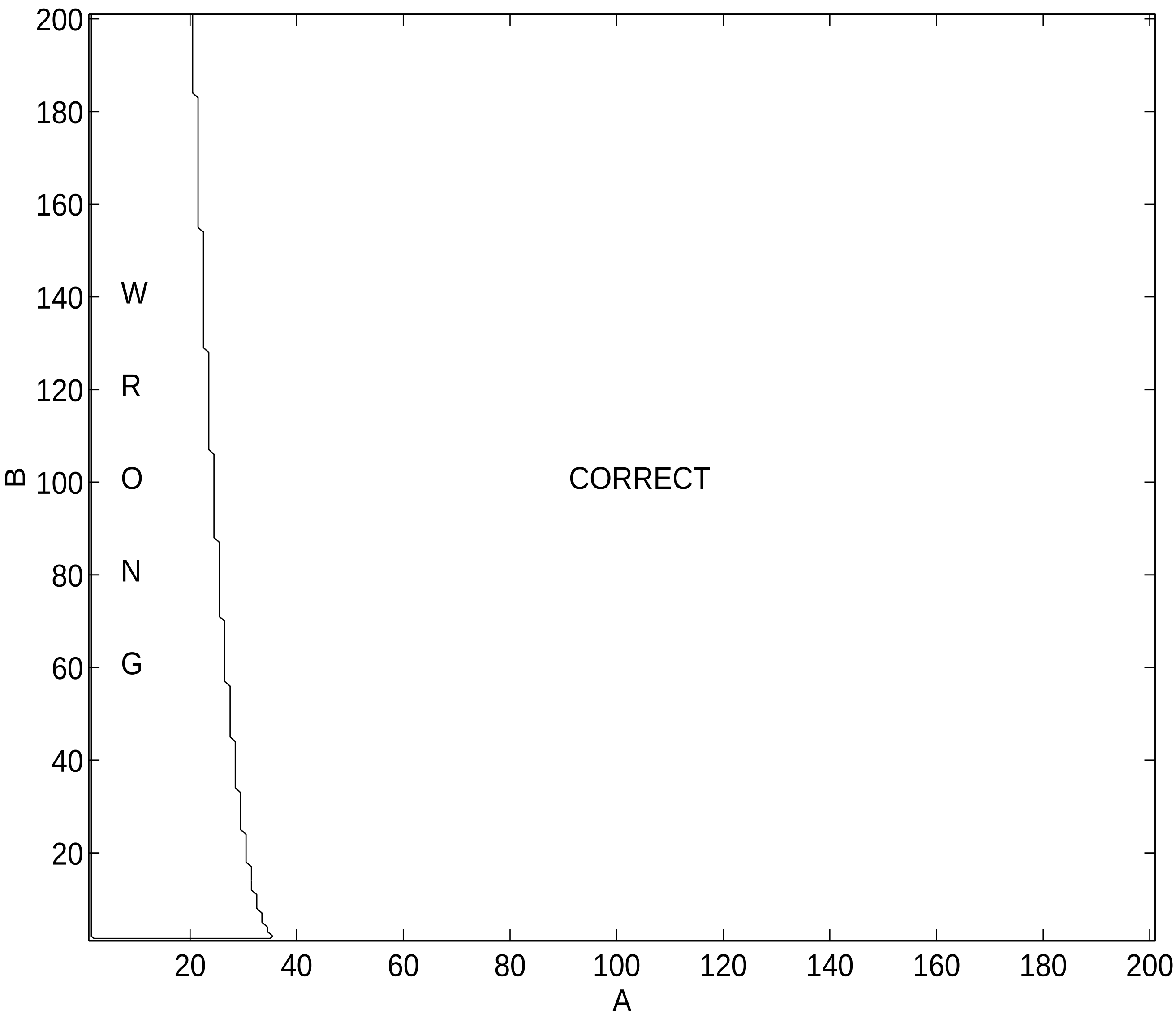

Figure 2: *Region* CORRECT *consists of pairs* (A, B) *for which* X_1 *is optimal as predicted by The Rule.*

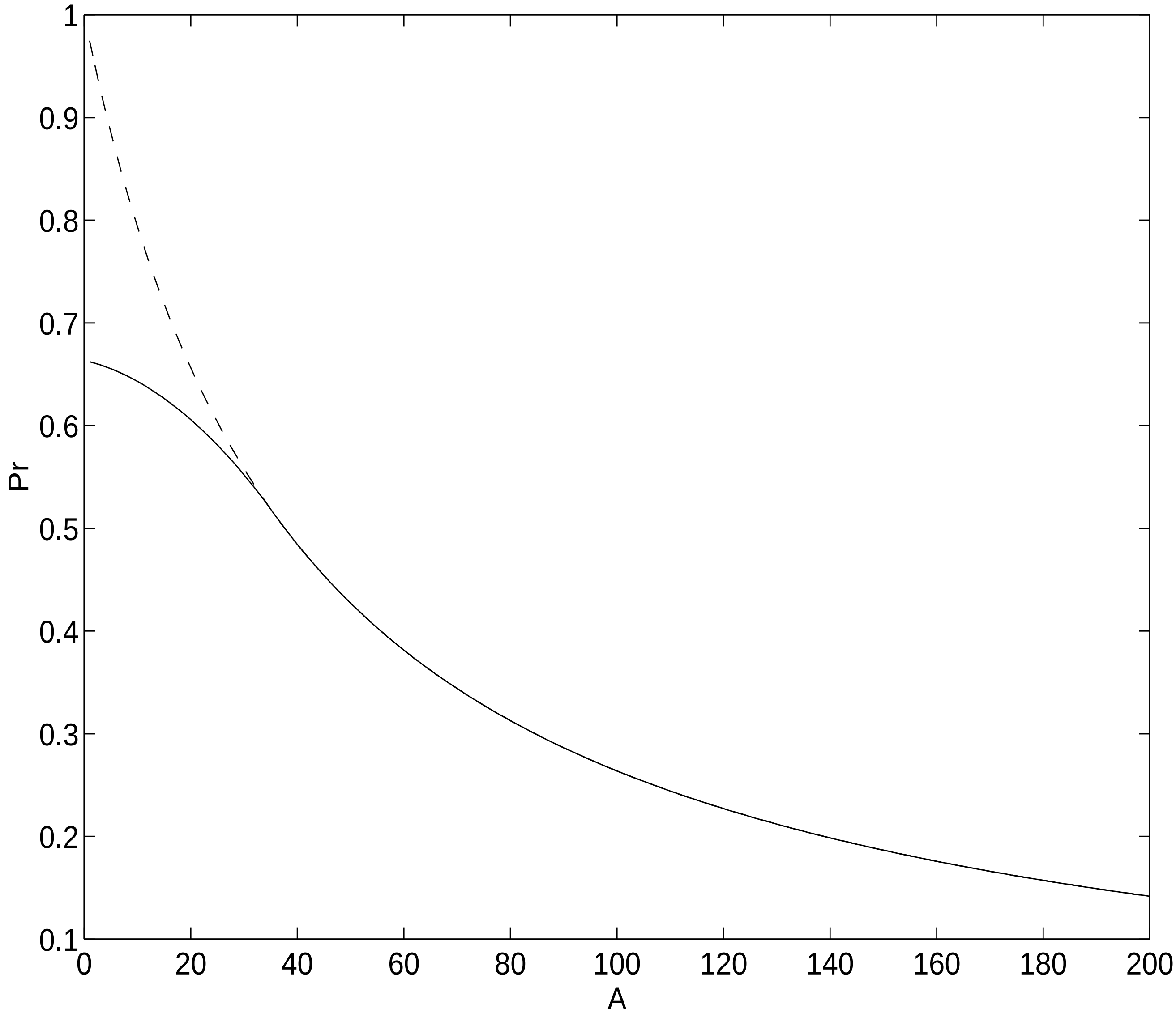

Figure 3: *Comparison of the approximation* $\bar{P}_F(A, B)$ *(dashed line) and the exact probability* $P_F(A, B)$ *(solid line) for* $k = 1$ *and* $B = 40$*.*

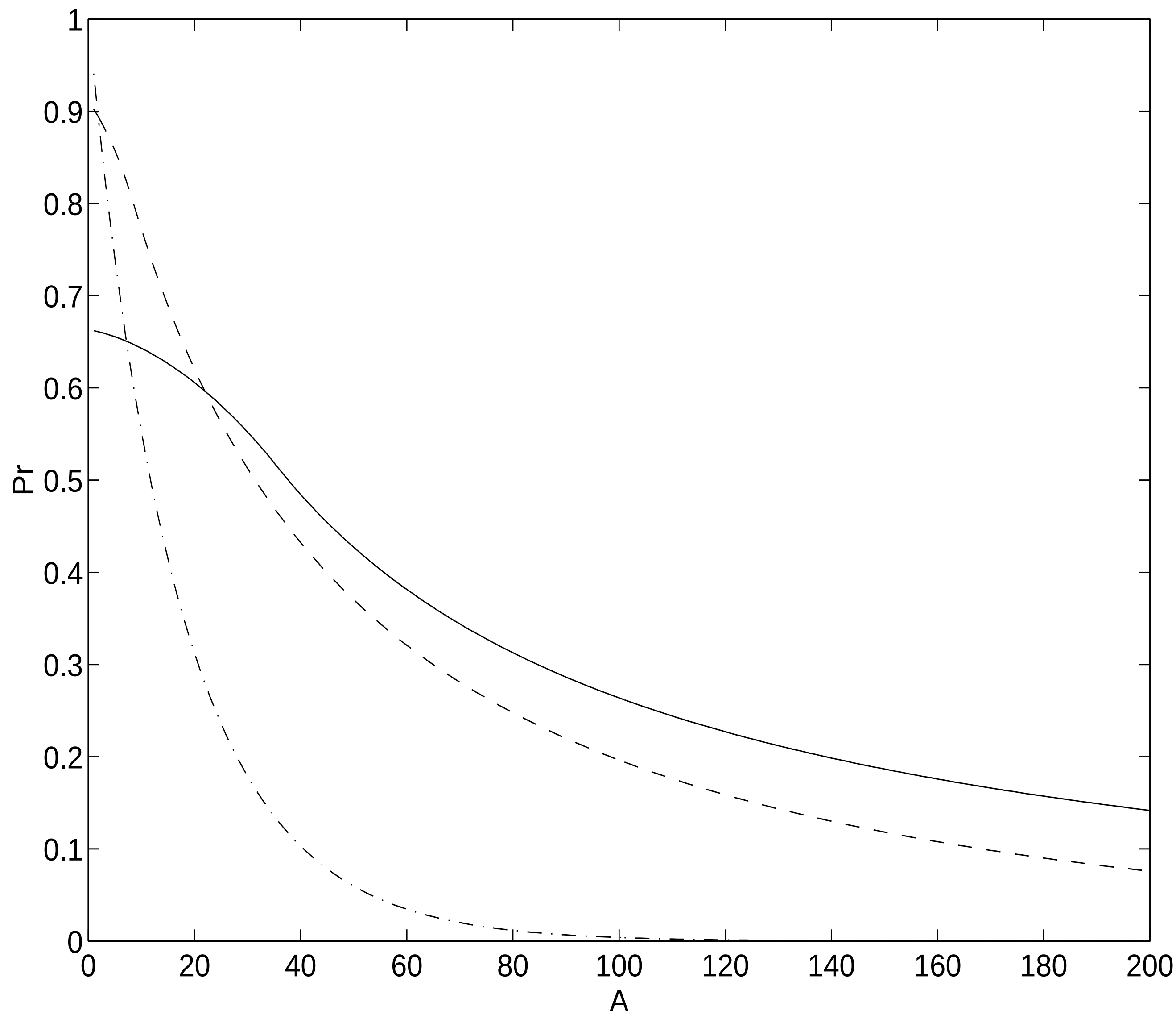

Figure 4: *The exact probabilities of achieving A before losing B in Roulette, for $B = 40$ and strategies corresponding to $k = 1$ (solid line), $k = 4$ (dashed line), and $k = 18$ (dashdot line).*

values of A they are in a good agreement with the ordering suggested by The Rule.

Remark 4 implies that since the expected value of a single play is negative the approximate probability $\bar{P}_F(A, B)$ of winning increases when the player increases the stakes. Hence, assuming that $B \geq 1.0$, the theoretical maximum is achieved at $s = B$. One should be however careful with the last conclusion because the overshooting effect may dominate for small B and A, and make the conclusion to The Rule incorrect. The effect of this type can be seen in Figure 4 for small values of A.

Another interesting feature of the Roulette game can be observed using approximation (1.17) of the expected time of the game. It is given by the limit

$$\lim_{A \to \infty} E[N] = -\frac{B}{E[X]}$$

assuming fixed B. The limit can be easily seen in graphs and admits an easy interpretation: very fast with an increase of A the ruin of the Gambler prevails very rapidly, and hence the expected time of the game is determined by the initial fortune of the Gambler and does not depend on any particular strategy. Yet it is surprising how much faster this upper bound for the expected time of the game is achieved with an increase of A while playing the Black Diamond or Red Diamond strategy in comparison with the Straight strategy.

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