Euler and the Genoese Lottery

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> His industry and genius have left a permanent impression in every field of mathematics; and although his contributions to the Theory of Probability relate to subjects of comparatively small importance, yet they will be found not unworthy of his own great powers and fame.
>
> Isaac Todhunter on Leonard Euler[1]

1 Introduction

Volume I.7 of Leonard Euler's *Opera Omnia* is probably most often consulted for a handful of papers on famous mathematical problems: the Bridges of Königsberg, the Knight's Tour, Magic Squares, the problem of Derangements and the Josephus Problem. These gems of recreational mathematics are to be found nearly buried among the more prevalent subject matter of the volume: some two dozen entries concerning probability theory and related subjects.

"Towards the middle of his life," wrote Louis Gustave du Pasquier, the editor of volume I.7, "Euler devoted a portion of his universal interest to the study of the theory of risk and ... to questions involving the calculus of probability [2, p. xxiii]." Alongside articles on observational error, mathematical statistics and the foundations of life insurance, volume I.7 contains eight memoirs and a fragment concerning probability theory on finite sample spaces. All of these are inspired by games of chance, be it the casino game *Pharaon*, the card game *Rencontre*, or the well-known Petersburg Problem. However, the greatest portion of Euler's writings on probability theory relate to the Genoese lottery.

Lotteries, the drawing of prizes "by lot," are as old as the written word. There are examples of goods being distributed by lot in ancient Israel and classical Rome but the more modern notion of a lottery as a form of gambling appears to have originated in Europe during the Renaissance. Florence is usually credited with holding, in 1530, the first public lottery in which chances were sold and cash prizes awarded, although the practice appears actually to have originated in Venice during the 1520s [3]. In any case, these early lotteries were of the type still used today for amateur fundraising events: a receipt for each ticket sold, bearing the player's name or some identifying number, was placed in an urn or other receptacle, and the winning tickets were drawn at random. The key development that would make large-scale lotteries possible was to be developed in a different Italian city-state, Genoa.

As early as the 11th century, executive power in the Genoese republic was vested in a small group of *Consigliori*, or Counselors. When Andrea Doria (1466-1560) ousted the French from Genoa in 1528 and assumed the mantle of Doge, he instituted government reforms which saw him sharing power with a council of five *Consigliori*. These were chosen by lot from a group of *Senatori*, or Senators, variously reported to number 100 [2] or 120 [3]. The residents of Genoa (where, according to du Pasquier, games of chance had been popular since time immemorial) took to placing bets on the outcome of the draw. Some time later, the number of candidates was reduced to 90, and the drawing was done by extracting numbered balls from an urn.

It was in 1620 that the idea of drawing five numbers from a list of 90 as a pure game of chance was abstracted from the election process. The credit for this is given to a local official named Benedetto Gentile. By 1643, the Genoese republic institutionalized such a lottery as a means of raising revenue for the state. In 1665, similar lotteries were established in Milan, Naples and Venice. Popes Innocent XI and XII tried unsuccessfully to reverse the trend, but eventually even the Vatican succumbed to the temptation of lottery revenues, and the Genoese lottery came to Rome in 1732.

Even today, the Italian government sponsors a state lottery called 'Lotto', which involves drawing five balls from a *ruota*, or wheel, containing balls numbered $1, 2, 3, \ldots, 90$. Fittingly, Genoa is one of ten cities in which weekly drawings are held.

2 The Royal Charge

In the middle of the 18th century, the Italian craze for lotteries swept the European continent. It was thus that an Italian named Roccolini approached Frederick the Great of Prussia in 1749 with a scheme to establish a Genoese-style lottery in Berlin. This was some eight years after Leonard Euler had come to Frederick's court from St. Petersburg. The king, as was his custom when mathematical matters were involved, called upon Euler for counsel.

Euler was already working on a royal assignment when Frederick's letter of 15 September 1749 arrived, along with a copy of Roccolini's proposal for the Berlin lottery. Euler set aside the earlier assignment – the design of a new hydraulic system for Frederick's summer palace, Sans-Souci – and two days later returned his analysis of the proposal to the king.

Roccolini's proposal involved the usual drawing of 5 numbers from a list of 90 and offered three principal ways for gamblers to place their wagers:

simple The player chooses one number between 1 and 90 and pays 8 *écus* to play. If the chosen number is among the 5 numbers drawn, the player wins 100 *écus*.

ambo The player chooses two numbers between 1 and 90 and pays 14 *gros* to play. If both the chosen numbers are among the 5 numbers drawn, the player wins 120 *écus*.

terno The player chooses three numbers between 1 and 90 and pays 15 *deniers* to play. If all three of the chosen numbers are among the 5 numbers drawn, the player wins 180 *écus*.

The *écu* was a French silver coin, worth 3 francs. It was divided into 24 *gros*, which were in turn each divided into 12 copper *deniers*. Thus 1 *écu* = 24 *gros* = 288 *deniers* and 1 *gros* = 12 *deniers*. Subsequently, all fractional parts of *écus* will be given as decimals, although Euler did not do this in his report of September 17.

It was also possible to place mixed bets, but apparently the proposal didn't specify the price of such tickets. Frederick's charge and Euler's response are reproduced in Volume IVA.6 of the *Opera Omnia* [5, pp. 316–320], but unfortunately Roccolini's proposal is not included.

Euler's analysis began with a calculation the fair price of each type of ticket, "according to the law of equality". In other words, he calculated the expected value of the payoff in each case. Table 1 summarizes his findings. In 1749, Euler determined the bank's profit using the measure

$$\frac{P-E}{E} \times 100\%,$$

where E is the expected value and P is price of a ticket. In his 1763 paper "Reflections on ... the Genoese Lottery", which we will discuss in the next section, he uses

$$\frac{P-E}{P} \times 100\%$$

to measure that bank's profit.

Type of bet	Probability of a win	Prize	Expected Value	Price of a Ticket	The Bank's Profit (1749)	The Bank's Profit (1763)
simple	$\frac{5}{90}$	100	5 5/9	8	44%	31%
ambo	$\frac{5\cdot 4}{90\cdot 89}$	120	0.2996	0.5833	95%	49%
terno	$\frac{5\cdot 4\cdot 3}{90\cdot 89\cdot 88}$	180	0.01532	0.05208	240%	71%

Table 1: Summary of Euler's Analysis

Euler's analysis continued with a discussion of the fairness of the bank's increasing profit margin. It is important to note here a significant difference between the Genoese lottery and most modern versions of the lottery. The Genoese lottery (and, for that matter the contemporary Italian Lotto) offers fixed-odds payoffs. That is, the player knows in advance how large a prize is at stake, irrespective of the number or distribution of tickets sold. In the long run, the bank is guaranteed to win, and to win big. However, if the state were to have a run of bad luck, and a relatively large number of major prizes were to be awarded in a drawing where relatively few tickets had been sold, the bank could be broken.

The danger of bankruptcy is avoided in lottery games played today in the USA and most other countries by adopting a version of the method of pari-mutuel betting used in horse racing. The term, literally meaning 'mutual wager', comes from two French words. A portion of each wager placed is set aside to cover the cost of running the lottery and the state's revenue, and the remainder is placed in a pool, to be paid out to the winners. In effect, players are betting against each other, with the state taking a cut of every wager.

In New York State's Lotto, for example, players choose 6 numbers between 1 and 51. They win a cash prize if 4, 5 or 6 of their chosen numbers match, and a free ticket in the next lottery if they match 3. Fifty-one cents of every one dollar ticket goes into a pari-mutuel pool, with a certain portion of that reserved for the jackpot, paid to players who match all 6 numbers in the draw. If there is no jackpot winner in a particular drawing, the jackpot rolls over, making the subsequent drawing even more attractive to players. Reliable estimates on the size of the jackpot are available before the drawing, but only at the close of betting is the exact payoff known.

In Euler's time, the technology needed to deliver the sort of up-to-the-minute information on ticket sales used in the calculation of pari-mutuel odds was not available. Therefore Euler noted in his letter to Frederick that it was entirely proper for the bank to offer relatively smaller payoffs for the riskier *ambo* and *terno* bets in order to insulate itself from calamity. He also observes that if the total number of bets is small, it would be undesirable to have many players choosing the same *ambo* or *terno* bets, although it's not clear what sort of practical bookkeeping procedures might have been instituted at that time to avoid these multiple bets.

We note that although Roccolini's proposal did not allow for a gambler to bet on all 5 numbers in the drawing, this sort of bet was permitted in other Genoese-style lotteries of the 18th century, including the lottery that was eventually instituted in Berlin in 1763. In fact, it was even possible in the French Royal lottery to bet on all five numbers **and** the order in which they were drawn. The payoff, at a million to one, made this game irresistibly attractive to the French populace, particularly the lower classes, despite the fact that the true odds are in excess of five billion to one. "There were occasions when mathematicians wrote learned articles demonstrating why people should not play the lottery at those odds," writes Katz [4, p. 598] in describing the French lottery, "but the only mathematicians to whom people paid attention were those who sold sure-fire methods for picking the winning numbers!"

Euler neither wrote cautionary epistles, nor did he hawk sure-fire winning strategies. Instead, the Genoese lottery inspired in him a series of mathematical articles concerning some of the trickier questions of combinatorics and probability theory to be tackled in the 18th century. The first glimmer of these deeper thoughts is his observation, towards the end of letter of September 17, that there is nothing sacred about the numbers 5 and 90; the Genoese lottery may be abstracted to any number n of numbered balls or tickets in the urn, and any number $t < n$ of such tokens chosen in the drawing. He closes by outlining for Frederick an alternate lottery scheme, involving $n = 100$, $t = 10$, where players bet on 1, 2, 3 or 4 numbers.

3 Reflections on a Singular Lottery

That the Genoese lottery captured Euler's mathematical imagination is evidenced by the entries he made in his notebook, known today as H5, which he probably filled between 1748 and 1750 [2, p. xxiv]. His first published article on the lottery did not appear until 1767. However, the Genoese lottery was only established in Berlin in 1763, and on March 10 of the same year, Euler delivered an address entitled "Reflections on a Singular Type of Lottery called the Genoese Lottery" [6] to the Berlin Academy. The text of the address was finally published in 1862, in Euler's *Opera Posthuma I*. The posthumous publication is reflected in the paper's relatively high number in the Eneström system for numbering Euler's works: E812.

Unlike Euler's later papers on the Genoese lottery, which are works of pure mathematics that set out to answer questions inspired by the lottery of little or no practical value, E812 is a work of applied mathematics. Euler proves little new mathematics, but instead applies elements of probability theory, interspersed with dashes of common sense and vaguely justified rules of thumb, in describing how one might go about designing a Genoese style lottery, determining, in particular, fair prize levels. This represents a shift in focus from Roccolini's proposal, where prizes were of a fixed value, and Euler determined the fair price of a ticket, comparing these to Roccolini's prices.

Euler's goal in the first portion of the paper is to calculate the probability $p_{k,i}$ that a player who bets on k numbers will in fact match i of them. The values depend on the parameters n and t, where the lottery consists of choosing t tokens at random from a collection numbered $1, 2, 3, \ldots, n$. A modern probability text would say that i has hypergeometric distribution, with parameters n, t and k, and thus

$$p_{k,i} = \frac{\binom{t}{i}\binom{n-t}{k-i}}{\binom{n}{k}}.$$

However, the hypergeometric distribution, the distribution of sampling without replacement, was not yet recognized as a standard random variable in Euler's time. Euler's analysis may indeed be one of the first treatments of this distibution in print. He did not give the $p_{k,i}$'s as above; in fact, he had not yet even developed a convenient notation for binomial coefficients (see section 5). Instead, Euler gave the $p_{k,i}$ in a form that was well-suited for efficient recursive calculation.

Euler gives a complete derivation of the desired probabilities. The method of proof – an implicit or 'socratic' induction – is a common one in Euler's writings: he solves the simplest cases in order, clearly and persuasively argued, until the pattern is clear to the reader. The "Reflections" paper is divided into sections which Euler calls Problems, some with corollaries or scholia. He considers the distribution of i in the case $k = 1$ in Problem 1, and then proceeds through the next three values of k in Problems 2-4. We summarize these in Table 2, where the kth column represents the results of the corresponding Problem.

$\mathbf{p_{k,i}}$	$k=1$	$k=2$	$k=3$	$k=4$
$i=0$	$\frac{n-t}{n}$	$\frac{(n-t)(n-t-1)}{n(n-1)}$	$\frac{(n-t)(n-t-1)(n-t-2)}{n(n-1)(n-2)}$	$\frac{(n-t)(n-t-1)(n-t-2)(n-t-3)}{n(n-1)(n-2)(n-3)}$
$i=1$	$\frac{t}{n}$	$\frac{2t(n-t)}{n(n-1)}$	$\frac{3t(n-t)(n-t-1)}{n(n-1)(n-2)}$	$\frac{4t(n-t)(n-t-1)(n-t-2)}{n(n-1)(n-2)(n-3)}$
$i=2$		$\frac{t(t-1)}{n(n-1)}$	$\frac{3t(t-1)(n-t)}{n(n-1)(n-2)}$	$\frac{6t(t-1)(n-t)(n-t-1)}{n(n-1)(n-2)(n-3)}$
$i=3$			$\frac{t(t-1)(t-2)}{n(n-1)(n-2)}$	$\frac{4t(t-1)(t-2)(n-t)}{n(n-1)(n-2)(n-3)}$
$i=4$				$\frac{t(t-1)(t-2)(t-3)}{n(n-1)(n-2)(n-3)}$

Table 2: Summary of Problems 1-4

Euler's reasoning is entirely elementary, and each Problem builds on the results of the previous one. Elements of the pattern come into focus quickly, particularly the presence of binomial coefficients. The details of Euler's solutions of Problems 1-4 further underline this identification, as the coefficient of every entry in each column, except the first and last, arises a sum of two coefficients in the previous column, thereby obeying precisely the same recursive formula as the entries in Pascal's triangle. Thus, we may express

$$p_{k,i} = \binom{k}{i} s_{k,i}.$$

In Problem 5, Euler asks for the general form of $p_{k,i}$. He introduces $r = n-t$ and writes out the cases $k = 1, 2, \ldots, 6$ in a new notation. Then $s_{k,i}$ is seen to be:

$$\frac{t(t-1)\cdots(t-i+1)r(r-1)\cdots(r-(k-i)+1)}{n(n-1)(n-2)\cdots(n-k+1)},$$

where $0 \leq i \leq k$, $1 \leq k \leq t \leq n$, and $k \leq r = n-t$.

Euler does not use this notation, nor does he explicitly give a formula for the general value of $s_{k,i}$. His notation is as follows:

$$\begin{array}{rcl} A^i & = & s_{1,i} \\ B^i & = & s_{2,i} \\ C^i & = & s_{3,i} \\ D^i & = & s_{4,i} \\ & \text{etc.} & \end{array}$$

In Corollary 1, Euler observes that the $s_{k,i}$ can be given recursively. We may give an explicit formulation as follows. Define $s_{0,0} = 1$; then for $k \geq 1$, and $0 \leq i < k$:

$$s_{k,i} = \frac{r - (k-i) + 1}{n - k + 1}\, s_{k-1,i}.$$

Also,

$$s_{k,k} = \frac{t - k + 1}{n - k + 1}\, s_{k-1,k-1}.$$

In Euler's notation, these formulas are given as follows:

$$B^2 = \frac{t-1}{n-1}A^1, \quad B^1 = \frac{r}{n-1}A^1, \quad B^0 = \frac{r-1}{n-1}A^0;$$

$$C^3 = \frac{t-2}{n-2}B^2, \quad C^2 = \frac{r}{n-2}B^2, \quad C^1 = \frac{r-1}{n-2}B^1, \quad C^0 = \frac{r-2}{n-2}B^0;$$

$$D^4 = \frac{t-3}{n-3}C^3, \quad D^3 = \frac{r}{n-3}C^3, \quad D^2 = \frac{r-1}{n-3}C^2, \quad D^1 = \frac{r-2}{n-3}C^1, \quad D^0 = \frac{r-3}{n-3}C^0;$$

etc.

In Problem 6, Euler turns his attention to finding the fair prize for a wager one *écu*. In this discussion, it is clear that Euler is considering a more elaborate lottery scheme than Roccolini's where, for example, a gambler playing *terno* must match all three of his $k = 3$ selections. In the "Reflections" paper, Euler will award a prize $F_{k,i}$ if i of the players' k numbers match the t numbers drawn, for **any** $0 < i \leq k$.

To do this, Euler simply chooses k positive numbers satisfying $\alpha_{k,1} + \alpha_{k,2} + \cdots + \alpha_{k,k} = 1$, and award prizes

$$F_{k,i} = \frac{\alpha_{k,i}}{p_{k,i}}.$$

Then the expected payoff for a ticket costing one *écu* is

$$\sum_{i=1}^{k} p_{k,i} F_{k,i} = \sum_{i=1}^{k} \alpha_{k,i} = 1.$$

Absent a notation uniform in k, the discussion of this simple point is surprisingly tedious, and is handled one case at a time, ending at $k = 5$. "It's not likely that we'd need to consider more than 5 numbers," Euler says, "as the prizes would be too exorbitant". Of course, it's precisely these 'exorbitant' prizes that make so many contemporary lotteries so very irresistible.

The $\alpha_{k,i}$s are not uniquely determined, unless $k = 1$. So for $k > 1$, Euler discusses three possible weighting schemes:

1. uniform weights

$$\alpha_{k,i} = \frac{1}{k}$$

2. binomial weights

$$\alpha_{k,i} = \frac{\binom{k}{i}}{2^k - 1}$$

3. modified binomial weights

$$\alpha_{k,i} = \frac{(k-i+1)\binom{k}{i}}{M_k} \quad \text{where} \quad M_k = \sum_{i=1}^{k}(k-i+1)\binom{k}{i}$$

Euler motivates methods **2** and **3** as progressively minimizing the impact of large prizes, corresponding to large values of i, on the bank. Curiously, he gives neither formulas nor even explanations for these weights, but simply tabulates the coefficients up to the case $k = 5$. I am grateful to Prof. Stephen Bloch of my department for help in solving the riddle of how the coefficients in method 3 were arrived at.

To illustrate these methods, let us compare in Table 3 the values of $\alpha_{5,i}$ in the three case.

Method	$i=1$	$i=2$	$i=3$	$i=4$	$i=5$
1	$\frac{1}{5}$	$\frac{1}{5}$	$\frac{1}{5}$	$\frac{1}{5}$	$\frac{1}{5}$
2	$\frac{5}{31}$	$\frac{10}{31}$	$\frac{10}{31}$	$\frac{5}{31}$	$\frac{1}{31}$
3	$\frac{25}{106}$	$\frac{40}{106}$	$\frac{30}{106}$	$\frac{10}{106}$	$\frac{1}{106}$

Table 3: coefficients for $k = 5$

The question Euler examines in Problem 7 is that of determining a fair rate of return for the bank. Euler mentions that the lottery must hold back something in order to cover its expenses, and observes that if the lottery is being used to finance important state needs, then further discounts on the prizes above and beyond what he recommends may be needed. He suggests that on prizes with only one number matched, the bank should hold back be no more than 10% of the revenue, as a greater discount "would be too obvious and disgust the participants". Thus, 90% of $F_{k,1}$ should be returned to the gambler for each k.

On the other hand, larger profit margins for larger values of i are both justified by the risk, and "will hardly be noticed, given that few people are in a position to calculate the fair value." Accordingly, he suggests returning 80%, 70%, 60% and 50% of $F_{k,i}$ to the player when $i = 2, 3, 4$ and 5, respectively. These profit margins of 10%, 20%, and so on, are far more modest than the 31%, 49% and 71% margins spelled out in Roccolini's proposal (see the final column of Table 1).

As with Problem 6, Euler's recommendations in Problem 7 do not have the force of mathematical proof behind them; they amount to little more than the recommendations and educated guesses of a respected scholar.

All that remains for Euler is to plug in values of n and t, and to calculate the size of the prizes under each of the three methods. In Problem 8 he does this in the canonical case of $n = 90$ and $t = 5$. Table 4 contains these recommended prizes.

In Problem 9, he uses the values $n = 100$ and $t = 9$, curiously close to the case $n = 100$ and $t = 10$ which he outlined for Frederick in 1749. Once again, he calculates the prize money under all 3 methods for each value of $k \leq 5$.

k	i	Method 1	Method 2	Method 3
1	1	16	16	16
2	2	160	106	64
	1	4	$5\frac{1}{2}$	$6\frac{1}{2}$
3	3	2,741	1,174	513
	2	36	47	41
	1	2	$2\frac{1}{2}$	$3\frac{1}{3}$
4	4	76,655	20,441	7,130
	3	526	561	391
	2	14	$22\frac{1}{2}$	24
	1	1	$1\frac{1}{4}$	$1\frac{3}{4}$
5	5	4,394,927	708,859	207,307
	4	12,409	10,007	5,853
	3	172	278	243
	2	7	$11\frac{1}{2}$	$13\frac{1}{2}$
	1	$\frac{3}{4}$	$\frac{1}{2}$	1

Table 4: Prizes for the canonical lottery

4 The Probability of Sequences

Euler's paper "On the Probability of Sequences in the Genoese Lottery" [7] was read to the Berlin Academy of Sciences in 1765 and published in the *Mémoires* of the academy two years later. In this memoir, Euler considers the probability that *sequences*, or runs of consecutive numbers, will appear among the numbers drawn in a Genoese style lottery. If, for example, the numbers 7, 8, 25, 26 and 27 are drawn, 7 and 8 constitute a *sequence of two* whereas 25, 26 and 27 constitute a *sequence of three*. It was not possible to bet on the occurrence of sequences in the Berlin lottery, so we must assume that this is simply a mathematical puzzle which Euler found appealing and worthy of his attention.

The structure of this paper, numbered 338 by Eneström, is similar to E812. A socratic induction is built up by solving increasingly complex cases. In Problems 1-4, Euler considers the probability of occurrence of the various patterns of sequences when $t = 2, 3, 4$ and 5, where n and t are as above. Although the cases $t = 2$ and 3 are relatively straightforward, it takes Euler almost 22 pages to complete all four Problems, by which time he has solved the case of the Genoese lottery.

In order even to state the results, we need further notation and the idea of a *species*. Euler denotes a sequence of i consecutive numbers by the shorthand (i). If the numbers $7, 8, 25, 26, 27$ are drawn, then he says the drawing has a species of $(3) + (2)$, as there is one sequences of 3 and one sequence of 2. If $t = 8$, then a drawing of $2, 3, 21, 22, 23, 57, 85, 86$ has a species of $1(3) + 2(2) + 1(1)$, since it has a sequence of 3, two sequences of 2 and one sequence of 1. In Table 5, we summarize results on the distribution of various species when $t = 5$ in both the general case and the canonical case $n = 90$.

Species	Probability, General Case	Probability, Genoese Lottery
(5)	$\frac{2\cdot3\cdot4\cdot5}{n(n-1)(n-2)(n-3)}$	$\frac{1}{511038}$
$(4)+(1)$	$2\cdot\frac{3\cdot4\cdot5(n-5)}{n(n-1)(n-2)(n-3)}$	$\frac{85}{511038}$
$(3)+(2)$	$2\cdot\frac{3\cdot4\cdot5(n-5)}{n(n-1)(n-2)(n-3)}$	$\frac{85}{511038}$
$(3)+2(1)$	$3\cdot\frac{4\cdot5(n-5)(n-6)}{n(n-1)(n-2)(n-3)}$	$\frac{3570}{511038}$
$2(2)+(1)$	$3\cdot\frac{4\cdot5(n-5)(n-6)}{n(n-1)(n-2)(n-3)}$	$\frac{3570}{511038}$
$(2)+3(1)$	$4\cdot\frac{5(n-5)(n-6)(n-7)}{n(n-1)(n-2)(n-3)}$	$\frac{98770}{511038}$
$5(1)$	$\frac{(n-5)(n-6)(n-7)(n-8)}{n(n-1)(n-2)(n-3)}$	$\frac{404957}{511038}$

Table 5: Distribution of species, $t = 5$

Over the course of 11 subsequent pages, Euler handles all 11 cases associated with $t = 6$. He then considers himself poised to state the solution to the general problem.

Given an arbitrary t, let the species σ of a draw be denoted

$$\alpha_1(l_1) + \alpha_2(l_2) + \cdots + \alpha_m(l_m).$$

Then necessarily

$$\sum_{j=1}^{m} \alpha_j l_j = t.$$

In addition, let

$$k = \sum_{j=1}^{m} \alpha_j.$$

Then the number of drawings which result in a species of σ is

$$\frac{(n-t+1)(n-t)(n-t-1)\ldots(n-t-k+2)}{\prod_{j=1}^{m} \alpha_j!}$$

All desired probabilities can be then be calculated by dividing this quantity by the total number of possible drawings, which is simply

$$\binom{n}{t}.$$

To completely solve the problem for a given t, then, one needs an exhaustive list of the various species, or *partitions*, associated with a natural number t. The partition function $p(t)$, which gives the the number of such partitions, is an essential component of a modern course in number theory. Euler lists the first 15 values of $p(t)$ in a corollary to the general result: 1, 2, 3, 5, 7, 11, 15, 22, 30, 42, 56, 77, 101, 135, and 176.

The partition function was studied extensively by Euler. He wrote papers on it in 1741 [8], 1750 [9] and 1769 [10]. In addition, he devoted a chapter of the first volume of his textbook *Introductio in analysin infinitorum*, published in 1748, to the study of $p(t)$. It is quite reasonable to speculate that it was this application in 1765 to probability theory that prompted Euler to revisit the subject of partitions shortly after his return to St. Petersburg in 1766.

5 Other Results

After his return to St. Petersburg, Euler wrote two more memoirs concerning matters of probability theory arising from the Genoese lottery. Both of these papers, numbered E600 [11] and E813 [12], concerned the number of distinct integers between 1 and n which would be drawn over the course of many repetitions of the lottery.

The first of these papers, "On the Solution of Difficult Questions in the Calculus of Probability" was presented to the Academy of St. Petersburg on October 8, 1781 and published posthumously in 1785. Euler begins with the observation that it would be convenient to have a special symbol for the binomial coefficient, and so he defines:

$$\left(\frac{p}{q}\right) = \frac{p(p-1)\cdots(p-q+1)}{q!}.$$

Actually, Euler used square brackets in place of the parentheses in the original published version of this paper, but in a variety of roughly contemporary papers, he used the above notation, and so the editors use the above notation consistently throughout the *Opera Omnia*.

The organization of this paper is different from the previous papers on the lottery. Euler begins by posing his Problems and providing the solutions immediately without proof. All proofs are presented in the second half of the paper.

The first major question is to determine the probability that, after d repetitions of a Genoese style lottery, all of the numbers from 1 to n will have been drawn at least once. Let t once again denote the number of tokens removed in a single drawing, and suppose that $td \geq n$ so that it is possible that all n tokens have been drawn. The size of the sample space is then

$$\Delta = \left(\begin{array}{c} n \\ t \end{array} \right)^d.$$

If we fix a subset of size k of the n numbers, then there are

$$\left(\begin{array}{c} n-k \\ t \end{array} \right)^d$$

ways in which the d drawings might occur, such that none of these k numbers appear. As there are $\left(\begin{array}{c} n \\ k \end{array} \right)$ such subsets of size k, let

$$E_k = \left(\begin{array}{c} n \\ k \end{array} \right) \left(\begin{array}{c} n-k \\ t \end{array} \right)^d$$

Then E_k is the number of ways in which $n-k$ **or fewer** distinct numbers may be chosen in the d drawings.

We note that a given two-element set $\{i, j\}$ will be counted twice in the consideration of E_1: once when we count the number of ways the drawings which exclude i may take place, and again when we consider the drawings that exclude j. Thus $\Delta - E_1$ undercounts the number of ways to choose all n tokens over the course of d drawings. To compensate for the two-element sets, a better approximation is $\Delta - E_1 + E_2$, but this overcounts the number of 3-element sets.

Clearly, this is a classic counting problem, familiar in the study of combinatorics, and the number of ways of drawing all n tokens is

$$\Sigma_0 = \Delta - E_1 + E_2 - E_3 + E_4 - \cdots,$$

where the alternating sum continues until E_r, where $n - r = t$. The probability that all n tokens are drawn is then

$$\frac{\Sigma_0}{\Delta} = 1 - \frac{E_1}{\Delta} + \frac{E_2}{\Delta} - \frac{E_3}{\Delta} + \frac{E_4}{\Delta} - \dots.$$

As an application, Euler considers the canonical case of the Genoese Lottery, with $d = 100$ drawings. Since $E_5/\Delta = 0$ to 4 decimal places, Euler needs only the first five terms of the alternating sum. The probability that all 90 numbers are chosen over the course of 100 drawings is 0.7410, although he reports it as 0.7411 due to round-off error. If $d = 200$, then the probability is 0.9990.

Having considered the number of ways in which all n numbers may be drawn in Problem 1, Euler considers the number of ways in which $n-1$ or $n-2$ of them may be drawn in Problems 2 and 3. With these 3 cases as the basis for his socratic induction, he states the general result: the number of ways in which the drawings can be held so that all but λ of the tickets are eventually drawn is

$$\Sigma_\lambda = \Delta - \sum_{k=1} (-1)^k \binom{\lambda + k - 1}{\lambda} E_{\lambda+k}.$$

To calculate the corresponding probability, we need only calculate Σ_λ/Δ.

The paper E813, entitled "Analysis of a Problem in the Calculus of Probability," was published in 1862. According to du Pasquier, the date of its composition is not known. Although it is discussed by du Pasquier after the paper E600 [2, p. xxvi], there appears to be no reason to assume that the papers were written in that order. It is shorter than the other other three papers we have considered, and reads more tersely when compared with the lucid technical prose of E812, E338 and E600. Perhaps it was a preliminary draft that Euler never managed to put into final form.

Consider the same situation of d repeated drawings where $dt \le n$ in this case. Then the number of distinct tokens drawn over the course of all the drawings is a discrete random variable U taking integer values between t and dt. Euler showas that the probability that U takes a value u in its range is

$$\frac{A_u (n-t)(n-t-1)\cdots(n-u+1)}{[n(n-1)\cdots(n-t+1)]^{d-1}}$$

for certain integer coefficients A_t, $A_{t+1}, \dots A_{dt}$. Here, the numerator is understood to be simply A_t when $u = t$.

The coefficients A_t, $A_{t+1}, \dots A_{dt}$ are not given in a closed form, but rather as the result of solving a polynomial equation of degree $(d-1)t$ in n. Perhaps Euler found this state of affairs to be somewhat unsatisfactory, and that this explains why he did not make it a priority to publish this paper.

6 Conclusion

Euler wrote four papers on the Genoese lottery, as well as a fifth one [13] on an entirely different type of lottery. The lottery described in this fifth paper was also one proposed to Frederick II, this time by a Dutchman named van Griethouse. There was a correspondence between Frederick and Euler in 1763 concerning this lottery proposal, quite similar to the one of 1749. This lottery was of the old style, as originally played in Venice and Florence, but with a new twist. Each player purchased not a single ticket, but a subscription to the same number in a series of repeated drawings. There is also a passing reference to this lottery in one of the applications contained in the paper E813.

All in all, Euler's writings on lotteries comprise but a tiny fraction of his total mathematical output. As Todhunter suggests in the opening quote, they may also be dismissed as being of little importance, especially when compared to his towering contributions to such fields as analysis and number theory.

This little backwater in Euler's great output begins with two simple requests for assistance from his royal patron. His duty was discharged in each case with the production of a short report for the king. Yet both problems stayed with him over the years, and apparently tickled a small fancy somewhere in his vast mathematical imagination. Therefore, this study may also serve to illuminate two admirable virtues of Euler's character: his sense of duty and devotion, and his ability to delight in mathematical recreation.

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