Theory of asymmetry-induced transport in a non-neutral plasma

D. L. Eggleston
Occidental College, Physics Department, Los Angeles, California 90041

T. M. O'Neil
University of California at San Diego, Department of Physics, La Jolla, California 92093

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Radial transport produced by static nonaxisymmetric fields is thought to limit the confinement of non-neutral plasmas and experiments with applied asymmetries have verified that such fields do produce transport. A theoretical model of such transport is presented which is appropriate for long, thin plasmas. The theory allows for asymmetries with nonzero frequency and includes the linear collective response to applied wall voltages. For the regime where the effective collision frequency is large, the asymmetry-induced radial particle flux is derived from the drift kinetic/Poisson equations including collisions. For low collision frequencies a heuristic derivation is given. In both regimes the resulting transport is dominated by particles that move in resonance with the asymmetry. Possible applications of the theory to several experiments are discussed. © *1999 American Institute of Physics.* [S1070-664X(99)01707-3]

I. INTRODUCTION

It has long been suspected that the ultimate confinement of long, thin non-neutral plasmas is limited by the presence of electric and magnetic fields that break the cylindrical symmetry of the trap. While experiments[1] at high neutral pressures agree very well with a transport theory[2] based on collisions with neutrals, at the lowest neutral pressures the confinement time is much lower than expected.[3] The anomalous transport increases with machine length[4] and decreases when experiments are performed in a device designed to minimize the field asymmetries.[5] While experiments with *applied* field asymmetries have verified that such fields produce radial transport, no connection to a transport theory has been made. A typical technique in these experiments is to apply asymmetric wall voltages to the various sectors of the confinement region and measure the resulting change in the transport.[6–8]

This paper presents a theory of the radial transport produced in a cylindrical non-neutral plasma by such applied asymmetric wall voltages. The wall voltages are allowed to have non-zero frequency so that the theory can apply to experiments with either static or nonstatic asymmetries. In contrast to the earlier phenomenological fluid theory of Fitzpatrick and Yu,[9] we allow the asymmetric potential to vary axially (as it does in most experiments) and base our theory on the drift kinetic equation with a collision operator. We have also included the plasma's collective response to the wall voltages and show that this can produce large changes in the transport flux.

Many of the basic notions involved in our theory were developed in early studies of radial transport in tandem mirrors,[10–14] where static asymmetric end cells produced radial grad-B drifts that largely determined the radial particle flux. A key prediction of both theories is that the resulting transport will be dominated by particles whose axial bounce motion and azimuthal drift motion causes them to move in resonance with the asymmetry. As these particles repeatedly encounter the asymmetry they take radial steps in the same direction, thus allowing them to diffuse more quickly than nonresonant particles. The form of the resulting radial particle flux depends on the relative size of an effective collision frequency ν_{eff} and the oscillation frequency ω_T of particles trapped in the asymmetry. When $\nu_{\mathrm{eff}} \gg \omega_T$, frequent collisions interrupt the trapped particle orbits and the basic radial step is the radial drift velocity times the time between collisions. Deviations from unperturbed orbits are small and a perturbation approach is appropriate. This is called the resonant plateau regime. When $\nu_{\mathrm{eff}} < \omega_T$, a trapped particle can complete at least one oscillation before a collision knocks it out of resonance. Now the basic radial step is the radial extent of the drift during a trapping oscillation and the orbits are fully nonlinear. A heuristic derivation of the resulting radial flux is often employed for this so-called banana regime.

The geometry of the non-neutral experiments is cylindrical with an axial magnetic field. The magnetic field is typically strong enough that the larmor radius is much smaller than any other scale length in the plasma and all relevant frequencies are small compared to the cyclotron frequency. Under these conditions the basic equations for an electron plasma ($q=-e$) are Poisson's equation,

$$\nabla^2 \phi = 4\pi e \int f dv, \tag{1}$$

the drift kinetic equation with a collision operator,

$$\frac{\partial f}{\partial t} + v \frac{\partial f}{\partial z} + \frac{e}{m} \frac{\partial \phi}{\partial z} \frac{\partial f}{\partial v} + \frac{c}{B} \hat{z} \times \nabla \phi \cdot \nabla f = C(f), \tag{2}$$

and the boundary conditions on the conducting walls. Here v is the axial velocity, $C(\ )$ is the approximate Fokker–Planck collision operator[15]

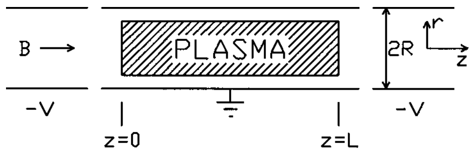

FIG. 1. Schematic of the plasma model used for this theory. The plasma is assumed to have flat ends and be of length L.

$$C(f)=\nu_{ee}\frac{\partial}{\partial v}\left[\overline{v}^2\frac{\partial f}{\partial v}+vf\right], \tag{3}$$

ν_{ee} is the 90° collision frequency, and $\overline{v}$ is the axial electron thermal velocity $\sqrt{T(r)/m}$. Expanding the gradients in Eq. (2) gives

$$\frac{\partial f}{\partial t}+v\frac{\partial f}{\partial z}+\frac{e}{m}\frac{\partial \phi}{\partial z}\frac{\partial f}{\partial v}+\frac{c}{B}\frac{1}{r}\frac{\partial \phi}{\partial r}\frac{\partial f}{\partial \theta}-\frac{c}{B}\frac{1}{r}\frac{\partial \phi}{\partial \theta}\frac{\partial f}{\partial r}=C(f). \tag{4}$$

To solve these equations we will linearize the guiding center distribution function f and the electrostatic potential ϕ and find the radial flux Γ in second order. Section II gives this derivation, and Sec. III discusses numerical methods for finding the potential produced by the wall voltages. Section IV discusses possible connections between the theory and various experiments.

II. DERIVATION OF THE TRANSPORT EQUATIONS

We take as our model a cylindrical plasma of length L with flat ends (see Fig. 1). The model thus ignores end effects and is most suitable for long, thin plasmas. This model allows us to replace the actual plasma by an infinitely long plasma with periodicity $2L$. It also allows us to linearize f and ϕ as follows:

$$\phi(r,\theta,z,t)=\phi_0(r)+\phi_1(r,\theta,z,t) \tag{5}$$

and

$$f(r,\theta,z,t)=f_0(r)+f_1(r,\theta,z,t). \tag{6}$$

Returning these to Eq. (4) and keeping only zeroth order terms gives

$$\frac{\partial f_0}{\partial t}+v\frac{\partial f_0}{\partial z}=C(f_0), \tag{7}$$

which has the well known solution

$$f_0(r)=\frac{n_0(r)}{\sqrt{2\pi\overline{v}^2}}\exp\left(-\frac{v^2}{2\overline{v}^2}\right). \tag{8}$$

Here $\overline{v}$ may also be a function of radius.

A. First order

Keeping terms of first order in Eq. (4) gives

$$\frac{\partial f_1}{\partial t}+v\frac{\partial f_1}{\partial z}+\frac{e}{m}\frac{\partial \phi_1}{\partial z}\frac{\partial f_0}{\partial v}+\frac{c}{B}\frac{1}{r}\frac{\partial \phi_0}{\partial r}\frac{\partial f_1}{\partial \theta}-\frac{c}{B}\frac{1}{r}\frac{\partial \phi_1}{\partial \theta}\frac{\partial f_0}{\partial r}$$
$$=C(f_1). \tag{9}$$

We now take advantage of the various periodicities in the model to write

$$\phi_1(r,\theta,z,t)=\sum_{n,l,\omega}\phi_{n,l,\omega}(r)\cdot\exp\left\{i\left(\frac{n\pi}{L}z+l\theta-\omega t\right)\right\} \tag{10}$$

and

$$f_1(r,\theta,z,t)=\sum_{n,l,\omega}f_{n,l,\omega}(r)\cdot\exp\left\{i\left(\frac{n\pi}{L}z+l\theta-\omega t\right)\right\}, \tag{11}$$

where the sums are over both negative and positive values. Note especially that ω can be positive or negative. A positive ω corresponds to an asymmetry that rotates in the same direction as the plasma column; a negative ω asymmetry rotates against the column. The Fourier mode amplitudes are given by

$$\phi_{n,l,\omega}(r)=\int_{-L}^{L}\frac{dz}{2L}\int_0^{2\pi}\frac{d\theta}{2\pi}\int_0^{\tau}\frac{dt}{\tau}$$
$$\times\exp\left\{i\left(\frac{n\pi}{L}z+l\theta-\omega t\right)\right\}\phi_1(r,\theta,z,t) \tag{12}$$

and similarly for $f_{n,l,\omega}$. Here τ is the duration of the experiment. Substituting in Eq. (9) and solving for $f_{n,l,\omega}$ we obtain

$$f_{n,l,\omega}(r)=\frac{\dfrac{cl}{rB}\dfrac{\partial f_0}{\partial r}-\dfrac{n\pi}{L}\dfrac{e}{m}\dfrac{\partial f_0}{\partial v}}{\dfrac{n\pi}{L}v+l\omega_R-\omega-i\nu_{\mathrm{eff}}}\phi_{n,l,\omega}(r). \tag{13}$$

Here we have noted that $(c/B)(1/r)(\partial\phi_0/\partial r)$ is the azimuthal $E\times B$ rotation frequency ω_R and have defined an effective collision frequency ν_{eff},

$$C(f_{n,l,\omega})\equiv-\nu_{\mathrm{eff}}\cdot f_{n,l,\omega}. \tag{14}$$

B. Second order

Since our interest is in radial transport we integrate Eq. (4) over z, θ, and v. Defining

$$N(r,t)=\int_{-L}^{L}\frac{dz}{2}\int_0^{2\pi}d\theta\int_{-\infty}^{\infty}dv\cdot f(r,\theta,z,t) \tag{15}$$

and noting $f(z=L)=f(z=-L)$ and $f(v=\pm\infty)=0$ we obtain

$$\frac{\partial N}{\partial t}+\frac{c}{rB}\int_{-L}^{L}\frac{dz}{2}\int_0^{2\pi}d\theta\int_{-\infty}^{\infty}dv\left[\frac{\partial\phi}{\partial r}\frac{\partial f}{\partial\theta}-\frac{\partial\phi}{\partial\theta}\frac{\partial f}{\partial r}\right]=0. \tag{16}$$

We note that the second term can be written as $(\partial/\partial r)[f(\partial\phi/\partial\theta)]-f(\partial/\partial r)(\partial\phi/\partial\theta)$ and after integrating by parts obtain

$$\frac{\partial N}{\partial t}-\frac{c}{B}\frac{1}{r}\frac{\partial}{\partial r}\int_{-L}^{L}\frac{dz}{2}\int_0^{2\pi}d\theta\int_{-\infty}^{\infty}dv\,f\frac{\partial\phi}{\partial\theta}=0. \tag{17}$$

Substituting in from Eqs. (5) and (6) gives

$$\frac{\partial N}{\partial t}=\frac{c}{B}\frac{1}{r}\frac{\partial}{\partial r}\int_{-L}^{L}\frac{dz}{2}\int_0^{2\pi}d\theta\int_{-\infty}^{\infty}dv\,f_1\frac{\partial\phi_1}{\partial\theta}. \tag{18}$$

From Eqs. (10) and (11),

$$f_1\frac{\partial\phi_1}{\partial\theta}=\left(\sum_{n,l,\omega}f_{n,l,\omega}(r)\cdot\exp\left\{i\left(\frac{n\pi}{L}z+l\theta-\omega t\right)\right\}\right)\cdot\left(\sum_{n',l',\omega'}il'\phi_{n',l',\omega'}(r)\cdot\exp\left\{i\left(\frac{n'\pi}{L}z+l'\theta-\omega' t\right)\right\}\right).\tag{19}$$

Eliminating $f_{n,l,\omega}$ using Eq. (13), the right-hand side of (19) becomes

$$\sum_{n,l,\omega}\sum_{n',l',\omega'}il'\phi_{n,l,\omega}\phi_{n',l',\omega'}\frac{\frac{cl}{rB}\frac{\partial f_0}{\partial r}-\frac{n\pi}{L}\frac{e}{m}\frac{\partial f_0}{\partial v}}{\frac{n\pi}{L}v+l\omega_R-\omega-i\nu_{\text{eff}}}\times\exp\left(i\left\{(n+n')\frac{\pi z}{L}+(l+l')\theta-(\omega+\omega')t\right\}\right).\tag{20}$$

Returning this expression to Eq. (18), we perform the z- and θ-integrals and also integrate over the duration of the experiment to obtain

$$\left\langle\frac{\partial N}{\partial t}\right\rangle=-\frac{c}{B}\frac{1}{r}\frac{\partial}{\partial r}2\pi L\int_{-\infty}^{\infty}dv\times\sum_{n,l,\omega}|\phi_{n,l,\omega}|^2 il\frac{\frac{cl}{rB}\frac{\partial f_0}{\partial r}-\frac{n\pi}{L}\frac{e}{m}\frac{\partial f_0}{\partial v}}{\frac{n\pi}{L}v+l\omega_R-\omega-i\nu_{\text{eff}}},\tag{21}$$

where $\langle\partial N/\partial t\rangle=(1/\tau)\int_0^\tau dt(\partial N/\partial t)$ and we have used the fact that $\phi_{-n,-l,-\omega}=\phi^*_{n,l,\omega}$. Multiplying top and bottom by the complex conjugate of the denominator and keeping the real part of the result (since the physical result must be real) gives

$$\left\langle\frac{\partial N}{\partial t}\right\rangle=2\pi L\frac{1}{r}\frac{\partial}{\partial r}r\sum_{n,l,\omega}\left|\frac{cl\phi_{n,l,\omega}}{rB}\right|^2\times\int_{-\infty}^{\infty}dv\frac{\nu_{\text{eff}}}{\left(\frac{n\pi}{L}v+l\omega_R-\omega\right)^2+\nu_{\text{eff}}^2}\times\left[\frac{\partial f_0}{\partial r}-\frac{n\pi}{L}\frac{e}{m}\frac{rB}{cl}\frac{\partial f_0}{\partial v}\right].\tag{22}$$

Noting that N is $2\pi L$ times the plasma density and recalling the particle continuity equation, we can identify the average radial particle flux as

$$\Gamma=-\sum_{n,l,\omega}\left|\frac{cl\phi_{n,l,\omega}}{rB}\right|^2\int_{-\infty}^{\infty}dv\frac{\nu_{\text{eff}}}{\left(\frac{n\pi}{L}v+l\omega_R-\omega\right)^2+\nu_{\text{eff}}^2}\times\left[\frac{\partial f_0}{\partial r}-\frac{n\pi}{L}\frac{e}{m}\frac{rB}{cl}\frac{\partial f_0}{\partial v}\right].\tag{23}$$

Note that although the flux involves an integral over all velocities this is strongly conditioned by the resonance function $\nu_{\text{eff}}\cdot[((n\pi/L)v+l\omega_R-\omega)^2+\nu_{\text{eff}}^2]^{-1}$. This function, which peaks at the velocity

$$v_{\text{res}}=\frac{L}{n\pi}(\omega-l\omega_R)\tag{24}$$

and has full-width at half-maximum $\Delta v=2L\nu_{\text{eff}}/n\pi$, shows that the flux tends to be dominated by particles that move resonantly with the asymmetry mode specified by n, l, and ω. If $\Delta v\ll\bar{v}$ (i.e., if the width of the resonance is small compared to variations in f_0), then

$$\frac{\nu_{\text{eff}}}{\left(\frac{n\pi}{L}v+l\omega_R-\omega\right)^2+\nu_{\text{eff}}^2}\rightarrow\frac{L}{|n|}\delta(v-v_{\text{res}}).\tag{25}$$

The velocity integral in Eq. (23) is now easily done. We obtain

$$\Gamma=-\sum_{n,l,\omega}\frac{L}{|n|}\left|\frac{cl\phi_{n,l,\omega}}{rB}\right|^2\left[\frac{\partial f_0}{\partial r}-\frac{n\pi}{L}\frac{e}{m}\frac{rB}{cl}\frac{\partial f_0}{\partial v}\right]_{v_{\text{res}}}.\tag{26}$$

If we plug in f_0 of the form of Eq. (8) this becomes

$$\Gamma=-\sum_{n,l,\omega}\frac{n_0}{\sqrt{2\pi\bar{v}^2}}\frac{L}{|n|}\left|\frac{cl\phi_{n,l,\omega}}{rB}\right|^2\left\{\frac{1}{n_0}\frac{dn_0}{dr}+\frac{1}{T}\frac{dT}{dr}\times\left(x^2-\frac{1}{2}\right)+\sqrt{2}\frac{n\pi}{L}\frac{r\omega_c}{l\bar{v}}x\right\}e^{-x^2},\tag{27}$$

where $x=v_{\text{res}}/\sqrt{2}\bar{v}$ and ω_c is the cyclotron frequency eB/mc.

It is worth noting several features of this solution. As is typical of plateau regime transport, the flux is independent of collision frequency and proportional to the square of the asymmetry amplitude. The plasma length L appears explicitly, but is also part of the variable x. Also hidden in this variable is the asymmetry frequency ω, and we note that x can be positive or negative as ω is greater than or less than ω_R. Thus, while static field asymmetries ($\omega=0$, $x<0$) move electrons radially outward, an appropriately chosen asymmetry ($\omega>\omega_R$, $x>0$) can move particles radially inward.

Equation (27) can be heuristically derived to within a numerical factor. Start from the relation[16]

$$\Gamma=-D\left(\frac{\partial f_0}{\partial r}\right)_H\Delta v,\tag{28}$$

where, for the case of a static asymmetry ($\omega=0$), the derivative is taken at constant H since the Hamiltonian is constant for a particle moving in a static field, the distribution function and diffusion coefficient are evaluated at the resonant velocity, and Δv is the width of the velocity resonance. Since the Hamiltonian for this plasma is $H=(p_z^2/2m)-e\phi_0+\mu B$, the Maxwellian distribution function can be written

$$f_0=\frac{n_0}{\sqrt{2\pi T/m}}\exp\left(-\frac{H+e\phi_0-\mu B}{T}\right).$$

Noting that n_0, T, and ϕ_0 are functions of r, it is straightforward to show that $(1/f_0)(\partial f_0/\partial r)_H$ reproduces the curly bracket of Eq. (27) for the case $\omega=0$. In order to include the $\omega\neq 0$ case, $(\partial f_0/\partial r)_H$ must be generalized to $(\partial f_0/\partial r)_{\bar{H}}$, where $\bar{H}=H-(\omega/l)P_\theta$ is the Hamiltonian in a frame rotating at frequency ω and P_θ is the single particle canonical angular momentum which in the guiding center approximation is equal to $-\frac{1}{2}m\omega_c r^2$.[17] The diffusion coefficient D is estimated as the average step size squared divided by the time between collisions: $D=(\Delta r)^2/\tau$. The average step size Δr is the radial $E\times B$ drift velocity $v_r=(cE_\theta/B)$ times the time between collisions. Since the relevant collisions occur at the enhanced rate ν_{eff} we obtain $D=(1/\nu_{\text{eff}})(cl\phi_{nl\omega}/rB)^2$. Finally the width of the velocity resonance Δv in the plateau regime can be obtained by taking the half-width of the velocity resonance function appearing in Eq. (23), $\Delta v=\nu_{\text{eff}}L/n\pi$. Plugging these estimates into Eq. (28), the factor of ν_{eff} cancels and the resulting flux is equal to the left-hand side of Eq. (27) divided by π.

As stated in the Introduction, the plateau regime corresponds to a collisionality regime where $\nu_{\text{eff}}\gg\omega_T$. For our case where the asymmetry varies in both z and θ, the trapping frequency is given by

$$\omega_T^2=\left\{\frac{e}{m}\left(\frac{n\pi}{L}\right)^2-\frac{cl^2}{rB}\frac{d\omega_R}{dr}\right\}\phi_{n,l,\omega}. \tag{29}$$

An estimate for ν_{eff} can be obtained by examining the form of Eqs. (3), (13), and (14). Because of the presence of a velocity resonance, the first term in Eq. (3) will dominate and we can estimate $\nu_{\text{eff}}\approx\nu_{ee}(\bar{v}/\Delta v)^2$. But the velocity resonance has half-width $\Delta v=\nu_{\text{eff}}L/n\pi$. Combining these we obtain

$$\nu_{\text{eff}}^3\approx\nu_{ee}\left(\frac{n\pi\bar{v}}{L}\right)^2=\nu_{ee}n^2\omega_b^2, \tag{30}$$

where ω_b is the axial bounce frequency $\pi\bar{v}/L$. Since ω_b is large compared to ν_{ee}, ν_{eff} will be larger than ν_{ee}. This is a reflection of the fact that only a small change in the velocity is necessary to knock a particle out of resonance. The collision time for this type of event is much less than for a ninety degree collision.

Similar heuristic arguments can be employed to obtain an approximate expression for Γ in the banana regime where $\omega_T\gg\nu_{\text{eff}}$. The basic radial step is now the width of the resonance island which may be estimated as $\Delta r\approx(v_r/\omega_T)=(cl\phi_{n,l,\omega}/rB\omega_T)$ and thus $D=\nu_{\text{eff}}(cl\phi_{n,l,\omega}/rB\omega_T)^2$. The width of the velocity resonance, which is broadened in the banana regime, is given by $\Delta v\approx(L/n\pi)\omega_T$. In this case the collision frequency does not cancel out and we obtain

$$\Gamma=-\sum_{n,l,\omega}\frac{\nu_{ee}\left(\frac{L}{n\pi}\right)^2\left(\frac{l\bar{v}}{r\omega_c}\right)^2\left(\frac{e\phi_{n,l,\omega}}{T}\right)^{1/2}}{\left\{1-\left(\frac{lL}{n\pi}\right)^2\frac{1}{r\omega_c}\frac{d\omega_R}{dr}\right\}^{3/2}}$$
$$\times\frac{n_0}{\sqrt{2\pi}}\left\{\frac{1}{n_0}\frac{dn_0}{dr}+\frac{1}{T}\frac{dT}{dr}\left(x^2-\frac{1}{2}\right)+\sqrt{2}\frac{n\pi}{L}\frac{r\omega_c}{l\bar{v}}x\right\}e^{-x^2}. \tag{31}$$

III. DETERMINATION OF THE ASYMMETRIC POTENTIAL IN THE PLASMA

In order to evaluate the flux we must determine the complex Fourier mode amplitudes $\phi_{n,l,\omega}(r)$ produced in the plasma by the applied wall potentials. For perturbed potentials of the form of Eq. (10), Poisson's Eq. (1) becomes [using Eq. (13) to eliminate $f_{n,l,\omega}$]

$$\left[\frac{1}{r}\frac{d}{dr}r\frac{d}{dr}-\frac{l^2}{r^2}-\left(\frac{n\pi}{L}\right)^2\right]\phi_{n,l,\omega}(r)=4\pi e\int dv\,\frac{\frac{cl}{rB}\frac{\partial f_0}{\partial r}-\frac{n\pi}{L}\frac{e}{m}\frac{\partial f_0}{\partial v}}{\frac{n\pi}{L}v+l\omega_R-\omega+i\nu_{\text{eff}}}\phi_{n,l,\omega}(r). \tag{32}$$

This equation must be solved subject to the conditions that $\phi_{n,l,\omega}$ is finite at $r=0$ and equal to the wall potential at $r=R$. Although analytical solutions exist[18] for special cases (e.g., constant density and temperature), a numerical solution is required for experimental density and temperature profiles. Fortunately, this is quickly and easily done using a modification of the ''shooting'' technique.

For Maxwellian f_0, the right hand side of Eq. (32) can be cast in terms of the plasma dispersion function[19] $Z(x)$ and its derivative $Z'(x)$ for which numerical codes exist. This relieves us of the task of numerically evaluating the integral. The result is[18]

$$\left[\frac{d^2}{dr^2}+\frac{1}{r}\frac{d}{dr}-\frac{l^2}{r^2}-\left(\frac{n\pi}{L}\right)^2\right]\phi_{n,l,\omega}=\left\{\frac{lL}{n\pi r\omega_c}Z(x)\frac{d}{dr}\left(\frac{\omega_p^2}{a}\right)-\left(\frac{\omega_p}{a}\right)^2\times Z'(x)\left[1+\frac{lLx}{n\pi r\omega_c}\frac{da}{dr}\right]\right\}\phi_{n,l,\omega}. \tag{33}$$

Here ω_p is the plasma frequency and $a=\sqrt{2}\bar{v}$. The radial derivatives on the left hand side are now written as second order central finite difference expressions.[20] Equation (33) then becomes

$$\frac{\phi_{j+1}-2\phi_j+\phi_{j-1}}{(\Delta r)^2}+\frac{1}{r_j}\frac{\phi_{j+1}-\phi_{j-1}}{2\Delta r}-\eta_j\phi_j=0. \tag{34}$$

Here we have suppressed the n,l,ω indices and have divided the space between $r=0$ and R into intervals of length Δr specified by the index j. The quantity η_j is defined as

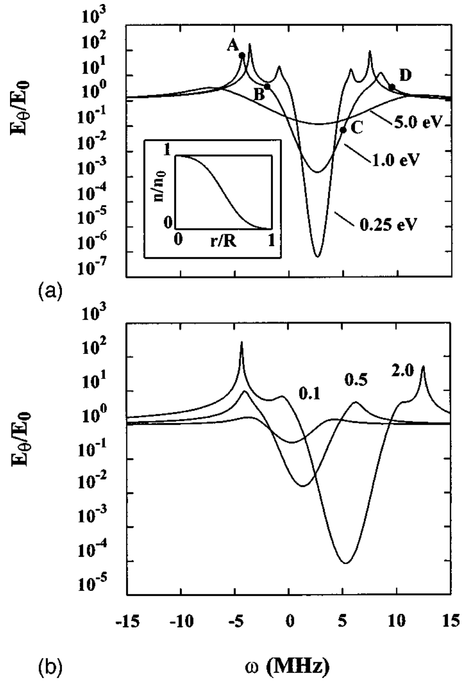

FIG. 2. $E_\theta(r=0)$ normalized to $E_0=E_\theta(r=R)$ vs asymmetry frequency for typical experimental parameters ($B=300$ G). (a) Peak density $n_0=10^7\ \mathrm{cm}^{-3}$ with temperature shown as a parameter. For comparison, $\omega_R(r=0)=3$ MHz. Inset shows the normalized radial density profile used for computation. (b) Temperature$=1$ eV with peak density (times $10^7\ \mathrm{cm}^{-3}$) shown as a parameter.

$$\eta_j=\frac{l^2}{r_j^2}+\left(\frac{n\pi}{L}\right)^2+\frac{lL}{n\pi r_j\omega_c}Z(x_j)\frac{d}{dr}\left(\frac{\omega_p^2}{a}\right)_j -\left(\frac{\omega_p}{a}\right)_j^2 Z'(x_j)\left[1+\frac{lLx}{n\pi r_j\omega_c}\frac{da_j}{dr}\right]. \tag{35}$$

Solving Eq. (34) for ϕ_{j+1} we obtain

$$\phi_{j+1}=\frac{2+(\Delta r)^2\eta_j}{1+\dfrac{\Delta r}{2r_j}}\phi_j-\frac{2r_j-\Delta r}{2r_j+\Delta r}\phi_{j-1}. \tag{36}$$

To use this generating equation we need the first two values of ϕ_j. We first note that solutions of the form $\phi_{n,l,\omega}=Ar^l$ satisfy Eq. (33) for small r. Since radial transport is produced only for cases where $l\neq 0$ (see the equations for Γ) we may take $\phi(r=0)=0$. For the second value, take any nonzero value. Since r_j and η_j are known for all j, we may now iterate (36) until we reach the wall radius and obtain ϕ_M where M is the index at the wall. However, $\phi_{n,l,\omega}(R)$ is known from Fourier analysis of the applied wall potentials. Thus if we multiply all the ϕ_j by $\phi_{n,l,\omega}(R)/\phi_M$ we have our solution.

The resulting solutions vary strongly with radius and asymmetry frequency ω, as well as several experimental parameters. Some indication of this is given in Figs. 2 and 3. Noting that Eq. (27) depends on $E_\theta=l\phi_{n,l,\omega}/r$, we plot in Fig. 2 the normalized magnitude of $E_\theta(r=0)$ vs ω for typical experimental parameters. Note that E_θ can vary by many orders of magnitude. This variation reflects typical plasma behaviors; the peaks occur at the frequencies of various standing waves and the strong dip near ω_R corresponds to Debye shielding. We emphasize, however, that there is continuous variation between these extreme cases and that an accurate determination of the transport flux depends sensitively on this calculation. Note also that, for a given asymmetry frequency, the field might be enhanced or diminished by these collective effects, depending on the details of the plasma parameters.

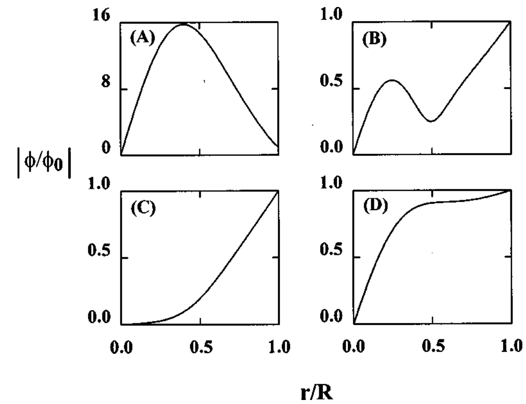

FIG. 3. Fourier mode amplitude $\phi(=\phi_{n,l,\omega})$ normalized to its value at the wall ϕ_0 vs radius. The labels A–D correspond to points indicated in Fig. 2(a). Graph A corresponds to a normal mode solution while C shows shielding behavior. Graphs B and D are intermediate cases. Note the difference in vertical scale for graph A.

This variation in the amplitude of E_θ is accompanied by strong variations in the radial dependence of $\phi_{n,l,\omega}$. A sampling of this variation is shown in Fig. 3, where we plot the normalized magnitude of $\phi_{n,l,\omega}$ vs r for the four frequencies indicated in Fig. 2(a) on the $T=1$ eV curve. Again the extreme cases of standing waves (3A) and shielding (3C) are shown along with two intermediate cases (3B, 3D).

IV. DISCUSSION

It is interesting to compare these results with previous and ongoing experimental work. The presence of asymmetry-induced transport in non-neutral plasmas was first suggested by the discovery[4] of confinement time scaling with $(L/B)^{-2}$. Comparing with our results, it is tempting to seize upon the $(L/\omega_{ce})^2\sim L^2/B^2$ in the leading factor of the banana regime flux given in Eq. (31) and contrast this with the L/B^2 for the plateau regime [Eq. (27)]. However, L is also hidden in the variable x, and the third term in brackets also contains ω_{ce}. Thus, without a knowledge of the spectrum of background asymmetries it is impossible to draw a firm conclusion.

Consistent with this theory, early experiments[6,21] found that standing waves could produce enhanced transport. These experiments also reported that modes rotating in the same direction as the plasma column but at a faster rate (i.e., $\omega>\omega_R$) produced inward transport. More recently, Huang and co-workers[8] and Anderegg and co-workers[22] have used an asymmetry with $\omega>\omega_R$ to balance the normal background transport and produce a steady-state plasma. The importance of standing waves in enhancing transport is also clear in this latter paper.

In order to study asymmetry-induced transport apart from such collective enhancements, Eggleston[23] has measured the confinement of low density ($<10^5\ \mathrm{cm}^{-3}$) electrons in a trap where a biased wire running along the axis of the trap replaces the plasma column. Under these conditions the variations in E_θ shown in Fig. 2 are essentially eliminated. The confinement time in this trap was found to have the same magnitude and $(L/B)^{-2}$ scaling as observed in the higher density experiments. Since the low density and higher temperature of this experiment give an electron–electron collision frequency ν_{ee} that is much lower than in the plasma experiments, it was argued that the transport (being the same in both) could not depend on collision frequency. While the plateau regime flux is independent of ν_{ee} it seemed unlikely that the transport could be in this regime due to the small value of ν_{ee}. However, this conundrum is somewhat softened by the fact that the transport regime depends on ν_{eff} rather than ν_{ee}, and the former has a much weaker dependence on density and temperature. Referring to Eq. (30), we see that although ν_{ee} goes like $n_0/T^{3/2}$, ν_{eff} goes like $n_0^{1/3}/T^{1/6}$. [Note that, in Eq. (30), n is the axial mode number, not the density n_0.] Thus the factor of 10^3 difference in ν_{ee} becomes a factor of 6 in ν_{eff}.

Several experiments have measured the amplitude-scaling of asymmetry-induced transport. In experiments with static (i.e., $\omega=0$) asymmetries, Notte and Fajans[7] observed a confinement time scaling $\tau \propto \phi^s$, with $s=1.72-2.14$, i.e., an amplitude scaling similar to the ϕ^2 dependence of Eq. (27). Interestingly, this experimental scaling was found to hold for wall voltages up to 40 V, well past the point where the plateau regime theory should apply. In contrast, more recent work with static asymmetries has found a robust linear scaling[24] while rotating-wall experiments[22] have given $s=0.7-1.1$. In low density experiments with time-varying asymmetries,[25] ϕ^2 scaling was observed for small amplitude asymmetries but at higher amplitudes the scaling was $\phi^{4/3}$. To our knowledge, no one has observed the banana-regime scaling of $\phi^{1/2}$.

Finally, Eggleston[26] has measured the flux produced by a single asymmetry mode (i.e., a single value of n and l) as a function of asymmetry frequency ω. The experiments show a resonance similar to that predicted by our theory and thus seem to confirm that the transport is dominated by resonant particles, but there are also important differences between the experiment and this theory. These results, along with a numerical comparison of the experimental and theoretical flux, will be presented in a subsequent paper.

From this discussion it should be clear that asymmetry-induced transport is far from understood. Several experimenters are currently studying this transport, but the results are not yet in agreement with each other or with any theory. While the theory presented in this paper can certainly stand further refinement (e.g., a more realistic treatment of particle motion at the ends of the plasma), we hope it will contribute to discussions of this phenomena.

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