Solution of Laplace's equation for the confining end potentials of a coaxial Malmberg–Penning trap

D. L. Eggleston
Department of Physics, Occidental College, Los Angeles, California 90041

Darrell F. Schroeter
Department of Physics, Reed College, Portland, Oregon 97202

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Laplace's equation is solved for the confining potentials at the ends of a coaxial form of the Malmberg–Penning charged particle trap. The solution gives insight into the confinement and dynamics of the trapped particles. The solution employs several mathematical methods that are often studied in isolation. The connections between these methods are illustrated by solving the problem in different ways. This process also produces several new integral and series identities.

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I. INTRODUCTION

The solution of Laplace's equation for various boundary conditions is a staple of intermediate and advanced courses in electrodynamics. The methods employed for these solutions have many applications in physics and engineering because it is common to specify potentials (rather than charge distributions) on conducting boundaries to produce the desired electric fields.

The problem addressed in this paper is motivated by the Malmberg–Penning charged particle trap, which is used for fundamental studies in various fields.[1] These traps usually employ a cylindrical geometry and use electric fields to provide axial confinement and a uniform axial magnetic field to provide radial confinement. The simplest device involves three conducting cylinders of equal radii placed in a line with a common axis. The middle cylinder is grounded, and the two end cylinders are held at a constant potential of the same sign as the trapped particles. The particles are trapped in the middle cylinder. The magnetic field is typically strong enough that the cyclotron radius is much smaller than any other length scale in the trap. Electric fields perpendicular to the magnetic field cause the center of the cyclotron orbit to drift with a velocity $(\vec{E}\times\vec{B})/B^2$.[2,3] In a cylindrically symmetric system, there are no azimuthal electric fields, and radial electric fields cause an azimuthal drift, which does not affect confinement.

Although particle confinement in these traps is good, it is not as good as expected.[4,5] It is believed that loss of confinement is due to azimuthal electric fields or magnetic field gradients associated with construction imperfections, and experiments with such fields have verified that they do produce radial transport.[6–8] However, a detailed understanding of this transport is still lacking, and there are serious discrepancies between experiment and theory.[9] It may be fruitful at this point to reconsider the particle dynamics to uncover the missing physics in the theory. Particle reflections at the ends of the trap, for example, are often modeled with an infinite potential at a fixed axial position (that is, specular reflection). Reality is more complicated but requires a detailed knowledge of the confining potentials at the end of the trap.

The problem addressed in this paper involves a modified version of the trap that adds a biased coaxial conductor.[5] The bias on this conductor produces a radial electric field that simulates the field produced by a column of plasma. Low density particles injected into this trap will have the same basic dynamics as in a plasma column, but the lowered density eliminates complex plasma phenomena (for example, plasma waves) that make it difficult to calculate the asymmetric fields in the plasma and the expected transport. The central conductor also provides improved experimental control by making it easy to adjust the radial electric field. However, the presence of this central conductor complicates the calculation of the confining end potentials. The goal of this paper is to find the potential inside the cylinders given the potential on the cylinders and the coaxial conductor. The trapped charged particle density is low enough so that the self-field of the particles is negligible compared to the field produced by the applied potentials, and thus Laplace's equation is applicable. The resulting solution employs several mathematical techniques, making it of pedagogical as well as practical interest.

We start by showing how the problem can be simplified by exploiting the axial symmetries. The symmetric part is solved by direct integration. The asymmetric part is approached using separation of variables, which requires the choice of a separation constant. In typical textbook presentations, an apparently arbitrary choice is made for the sign of this separation constant, and the solution involves developing the techniques necessary to solve the resulting ordinary differential equations. The consequences of making the opposite choice for the sign of the separation constant are not usually explored. We show that depending on the choice of the sign of the separation constant, we obtain either an integral or a series solution, each involving different techniques. We connect these two solutions by showing that the series solution can be obtained from the integral solution by using contour integration. In the process we discover several new integral and series identities.

II. INTEGRAL SOLUTION

An idealized version of one end of the trap is represented in Fig. 1. Two semi-infinite conducting cylinders of radius b are joined at $z=0$. The cylinder on the left is at the potential ϕ_b, while the cylinder on the right is grounded. There is also an infinitely long conductor of radius a coaxial with the cylinders and held at the potential ϕ_a. We seek the potential $\phi(r,z)$ for all z and for $a\leq r\leq b$.

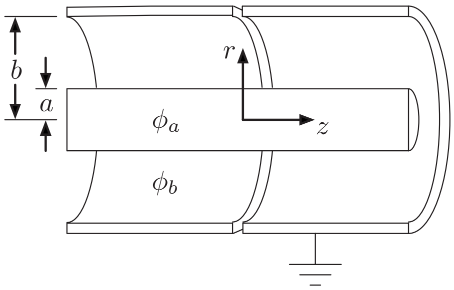

Fig. 1. Schematic of the potentials at one end of the trap. The outer cylinder at radius b is split into two halves, the left half at potential ϕ_b and the right half grounded. The central coaxial conductor of radius a is held at potential ϕ_a.

Our approach is to break the problem into its symmetric and antisymmetric parts in z, solve each part individually, and then reassemble the parts to describe the actual device. The symmetric part consists of an inner conductor at potential ϕ_a and both halves of the outer conductor at $\phi_b/2$, and the antisymmetric part has the inner conductor grounded and the outer conductor at $(\phi_b/2)\mathrm{sgn}(z)$, where $\mathrm{sgn}(z)=1$ for $z>0$ and $\mathrm{sgn}(z)=-1$ for $z<0$. If ϕ_1 is the solution to the symmetric part and ϕ_2 is the solution to the antisymmetric part, then $\phi_1-\phi_2$ is the solution to the original problem.

We start with Laplace's equation in cylindrical coordinates for azimuthal symmetry

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \phi}{\partial r}\right)+\frac{\partial^2\phi}{\partial z^2}=0. \tag{1}$$

The symmetric part of the problem has no z-dependence, and thus for this part we have $d/dr(r(d\phi_1/dr))=0$, which can be integrated to give $\phi_1(r)=A_0+B_0\ln(r)$. The application of the boundary conditions $\phi_1(a)=\phi_a$ and $\phi_1(b)=\phi_b/2$ allows us to determine A_0 and B_0. After some algebra we obtain

$$\phi_1(r)=\frac{\phi_b}{2}\frac{\ln(r/a)}{\ln(b/a)}+\phi_a\frac{\ln(b/r)}{\ln(b/a)}. \tag{2}$$

For the antisymmetric part of the problem, we seek a solution of the form $\phi_2(r,z)=R(r)Z(z)$. Equation (1) becomes

$$\frac{1}{R}\left[\frac{d^2R}{dr^2}+\frac{1}{r}\frac{dR}{dr}\right]=-\frac{1}{Z}\frac{d^2Z}{dz^2}. \tag{3}$$

For a positive separation constant k^2, the equation for $R(r)$ is the modified Bessel equation of order zero

$$r^2\frac{d^2R}{dr^2}+r\frac{dR}{dr}-k^2r^2R=0, \tag{4}$$

with the solution

$$R(r)=\alpha(k)I_0(kr)+\beta(k)K_0(kr). \tag{5}$$

The solution for $Z(z)$ is

$$Z(z)=\gamma(k)e^{ikz}+\delta(k)e^{-ikz}. \tag{6}$$

Here I_0 and K_0 are modified Bessel functions of order zero, and α, β, γ, and δ are k-dependent constants. Because of the antisymmetry of the boundary conditions, we know that $Z(z)=-Z(-z)$. We apply this condition to Eq. (6) and find that $\gamma=-\delta$, which allows us to combine the complex exponentials using the relation $e^{ikz}-e^{-ikz}=2i\sin(kz)$. Because there are no boundaries in the z-direction, the general solution takes the form of a Fourier transform[10,11] rather than a series. After combining some of the constants, we obtain

$$\phi_2(r,z)=\int_0^\infty [A(k)I_0(kr)+B(k)K_0(kr)]\sin(kz)dk, \tag{7}$$

where $A(k)$ and $B(k)$ are constants to be determined. Note that using the Eq. (5) solution for $k=0$ is not problematic because the small argument limits for I_0 and K_0 reproduce the logarithmic solution of Eq. (4) for $k=0$.

We now apply the boundary conditions. First, the potential on the center conductor for this part of the problem is zero. Because there is no z-dependence, it must be that

$$A(k)I_0(ka)+B(k)K_0(ka)=0 \tag{8}$$

for all k, and thus

$$B(k)=-A(k)I_0(ka)/K_0(ka). \tag{9}$$

The general solution can thus be reduced to the form

$$\phi_2(r,z)=\int_0^\infty A(k)F_0(kr)\sin(kz)dk, \tag{10}$$

where we have defined

$$F_0(kr)=I_0(kr)K_0(ka)-I_0(ka)K_0(kr) \tag{11}$$

and have absorbed a factor of $1/K_0(ka)$ into the constant $A(k)$.

The boundary condition at $r=b$ gives

$$\int_0^\infty A(k)F_0(kb)\sin(kz)dk=\frac{\phi_b}{2}\mathrm{sgn}(z). \tag{12}$$

We now write $\mathrm{sgn}(z)$ as a Fourier transform[12] and then as a sine transform

$$\mathrm{sgn}(z)=\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{1}{ik}e^{ikz}dk=\frac{1}{\pi}\int_0^\infty\frac{1}{ik}[e^{ikz}-e^{-ikz}]dk$$
$$=\int_0^\infty\frac{2}{\pi k}\sin(kz)dk. \tag{13}$$

We employ this last result in Eq. (12) and obtain

$$A(k)=\frac{\phi_b}{\pi k F_0(kb)} \tag{14}$$

and thus

$$\phi_2(r,z)=\frac{\phi_b}{\pi}\int_0^\infty\frac{F_0(kr)}{kF_0(kb)}\sin(kz)dk. \tag{15}$$

A contour plot of the solution using representative values of the parameters ($a/b=0.3$ and $\phi_a/\phi_b=0.25$) is shown in Fig. 2. The numerical integration was done using MATHEMATICA.[13] Figure 2(a) shows contours of ϕ_2/ϕ_b; the full solution $(\phi_1-\phi_2)/\phi_b$ is shown in Fig. 2(b). The radial and axial coordinates have been scaled by the wall radius b.

Although a detailed analysis of the effect of this potential on particle dynamics is beyond the scope of this paper, a number of significant insights may immediately be gleaned. By construction, all of the z-variation in the potential is in the function ϕ_2, and from Fig. 2(a) we see that this variation occurs in the neighborhood of $z=0$. Indeed, if z is large

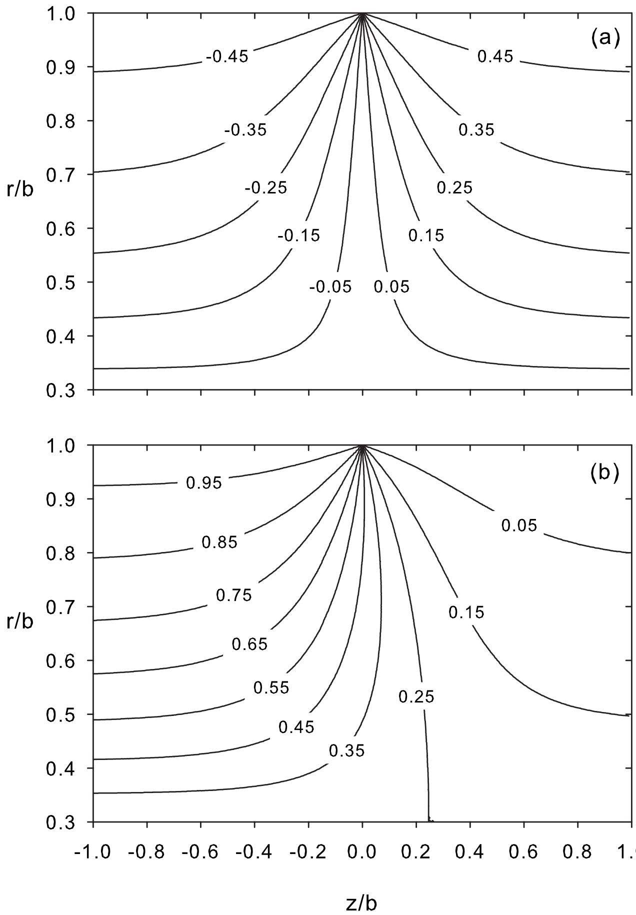

Fig. 2. Contour plots of the scaled potential for $a/b=0.3$ and $\phi_a/\phi_b=0.25$. (a) Contours of the asymmetric part of the solution ϕ_2/ϕ_b. (b) Contours of the full solution $(\phi_1-\phi_2)/\phi_b$. The radial and axial coordinates have been scaled by the wall radius b.

enough, there is no z-variation in ϕ_2 (see Sec. III). For the large magnetic fields employed in these traps, the variation in the particle's radial position associated with its cyclotron motion is negligible, and, as noted in Sec. I, radial electric fields do not change the radial position of a particle. If a particle of charge q starts at position z_0 with kinetic energy K, it will reflect at position z_r given by $q\phi(r,z_r)=q\phi(r,z_0)+K$ or $K/q=\phi_2(r,z_0)-\phi_2(r,z_r)\equiv\Delta\phi_2$. The reflection point will thus be a function of the particle's kinetic energy, with higher energy particles penetrating further into the confining potential. Also note from Fig. 2(a) that the maximum value of $\Delta\phi_2/\phi_b$ is a function of the radius, varying from one at $r/b=1$ to zero at $r/b=0.3$. Therefore, there will not be enough potential variation at small radii to confine the particles. This conclusion is consistent with the experimentally observed radial variation of the trapped particle density.[14]

III. COMMENTS ON THE INTEGRAL SOLUTION

As a check, we note that the z-dependence of $\phi_2(r,z)$ occurs only in $\sin(kz)$, and thus the solution is odd in z as required by the boundary conditions. Also the potential ϕ_2 at $z=0$ is zero, independent of r.

The behavior of the solution for large values of $|z|$ is not obvious, but can be obtained as follows. For all but the smallest values of k, the $\sin(kz)$ term in Eq. (15) oscillates rapidly with k compared to the variations in $F_0(kr)/kF_0(kb)$. Thus, these large k contributions will integrate to zero. For small k, we can use the small argument approximations for the Bessel functions[15] in F_0: $I_0(z)\approx 1$ and $K_0(z)\approx -\ln(z)$. Then the k-dependence cancels,

$$\frac{F_0(kr)}{F_0(kb)} \approx \frac{-\ln(ka)+\ln(kr)}{-\ln(ka)+\ln(kb)} = \frac{\ln\left(\dfrac{r}{a}\right)}{\ln\left(\dfrac{b}{a}\right)} \tag{16}$$

and

$$\int_0^\infty \frac{F_0(kr)}{kF_0(kb)}\sin(kz)dk \approx \frac{\ln\left(\dfrac{r}{a}\right)}{\ln\left(\dfrac{b}{a}\right)}\int_0^\infty \frac{\sin(kz)}{k}dk = \frac{\ln\left(\dfrac{r}{a}\right)}{\ln\left(\dfrac{b}{a}\right)}\left(\frac{\pi}{2}\right)\mathrm{sgn}(z), \tag{17}$$

where we have used standard integral tables[16,17] to evaluate the integral. By using this result in Eq. (15), we obtain

$$\phi_2(r,z\to\pm\infty) = \pm\frac{\phi_b}{2}\frac{\ln(r/a)}{\ln(b/a)}. \tag{18}$$

These large $|z|$ results can be understood intuitively. We expect the influence of the change in the $r=b$ boundary condition to diminish as we move away from $z=0$, and the potential will thus become z-independent. As with the symmetric case, Laplace's equation can then be directly integrated to give a solution of the form $\phi_2(r)=A_0+B_0\ln r$. Applying the appropriate boundary conditions at $r=a$ and $r=b$ gives Eq. (18).

IV. SERIES SOLUTION

As noted, our intuition leads us to expect that the influence of the z-dependent boundary conditions will diminish further from the boundary. This decrease is more easily seen when the solution is in the form of a Fourier–Bessel series.[18,19] We now show that such an approach is possible but requires an integral identity that has not appeared in the literature.

As before we start with Laplace's equation and now choose the separation constant to be $-k^2$ rather than k^2. We then obtain Bessel's equation of order zero for the r-dependence

$$r^2\frac{d^2R}{dr^2} + r\frac{dR}{dr} + k^2r^2R = 0, \tag{19}$$

with solutions $J_0(kr)$ and $Y_0(kr)$, where J_0 and Y_0 are Bessel functions of the first and second kind, respectively.[15] The z-dependence involves real exponentials $e^{\pm kz}$, one of which is discarded so that the potential remains finite for large $|z|$. We also know intuitively that for $z\to\pm\infty$ the potential is given by Eq. (18). Thus the solution has the form

$$\phi_2(r,z) = \pm \frac{\phi_b}{2}\frac{\ln(r/a)}{\ln(b/a)} + \sum_k [B_k J_0(kr) + C_k Y_0(kr)]e^{\mp kz}, \tag{20}$$

where the sum is over all allowed k-values and B_k and C_k are constants. Here and in the following, the upper (lower) sign applies for $z>0$ ($z<0$). By applying the boundary condition $\phi(a,z)=0$, we see that $B_k J_0(ka)+C_k Y_0(ka)=0$ for each k. We eliminate C_k, form a new constant $A_k=B_k/Y_0(ka)$, and write Eq. (20) as

$$\phi(r,z) = \pm \frac{\phi_b}{2}\frac{\ln(r/a)}{\ln(b/a)} + \sum_k A_k \tilde{F}_0(kr)e^{\mp kz}, \tag{21}$$

where

$$\tilde{F}_0(kr) = J_0(kr)Y_0(ka) - J_0(ka)Y_0(kr). \tag{22}$$

We next apply the boundary condition $\phi(b,z) =(\phi_b/2)\mathrm{sgn}(z)$. Because this condition is satisfied by the first term in Eq. (21), the k-values are restricted to those satisfying $\tilde{F}_0(kb)=0$. We denote these values as k_m, where $m=0,1,2,\ldots$. The solution thus becomes

$$\phi_2(r,z) = \pm \frac{\phi_b}{2}\frac{\ln(r/a)}{\ln(b/a)} + \sum_{m=0}^{\infty} A_m \tilde{F}_0(k_m r)e^{\mp k_m z}. \tag{23}$$

It remains to determine the constants A_m. To do so, we note that the antisymmetry of the boundary conditions implies that $\phi_2(r,z=0)=0$. By using this condition in Eq. (23), we obtain

$$\sum_{m=0}^{\infty} A_m \tilde{F}_0(k_m r) = \mp \frac{\phi_b}{2}\frac{\ln(r/a)}{\ln(b/a)}. \tag{24}$$

We multiply both sides of Eq. (24) by $r\tilde{F}_0(k_n r)dr$ and integrate from a to b. To evaluate the left-hand side, we use the result given in Appendix A,

$$\int_a^b r\tilde{F}_0(k_m r)\tilde{F}_0(k_n r)dr = \frac{\delta_{nm}}{2}h(k_m,a,b), \tag{25}$$

where

$$h(k_m,a,b) = b^2[\tilde{F}_1(k_m b)]^2 - a^2[\tilde{F}_1(k_m a)]^2, \tag{26}$$

$$\tilde{F}_1(kr) = J_1(kr)Y_0(ka) - J_0(ka)Y_1(kr), \tag{27}$$

and δ_{mn} is the Kronecker delta. We thus obtain

$$A_m = \frac{\mp \phi_b}{h(k_m,a,b)\ln(b/a)}\int_a^b r\tilde{F}_0(k_m r)\ln(r/a)dr. \tag{28}$$

The integral on the right-hand side of Eq. (28) does not appear in the usual integral tables[15–17] nor is recognized by MATHEMATICA.[13] In Appendix B it is shown that

$$\int_a^b r\tilde{F}_0(k_m r)\ln(r/a)dr = \frac{b}{k_m}\ln(b/a)\tilde{F}_1(k_m b), \tag{29}$$

where

$$\tilde{F}_1(k_m b) = J_1(k_m b)Y_0(k_m a) - J_0(k_m a)Y_1(k_m b). \tag{30}$$

The solution thus becomes

$$\phi_2(r,z) = \pm \phi_b\left[\frac{1}{2}\frac{\ln(r/a)}{\ln(b/a)} - \sum_{m=0}^{\infty}\frac{b}{k_m h(k_m,a,b)}\tilde{F}_1(k_m b)\tilde{F}_0(k_m r)e^{\mp k_m z}\right]. \tag{31}$$

Because $\phi_2(r,0)=0$, we also obtain the series identity

$$\sum_{m=0}^{\infty}\frac{b}{k_m h(k_m,a,b)}\tilde{F}_1(k_m b)\tilde{F}_0(k_m r) = \frac{1}{2}\frac{\ln(r/a)}{\ln(b/a)}. \tag{32}$$

If we compare the solution in Eq. (31) with the result in Eq. (15), we obtain the integral identity

$$\int_0^{\infty}\frac{F_0(kr)}{kF_0(kb)}\sin(kz)dk = \pm \pi\left[\frac{1}{2}\frac{\ln(r/a)}{\ln(b/a)} - \sum_{m=0}^{\infty}\frac{b}{k_m h(k_m,a,b)}\tilde{F}_1(k_m b)\tilde{F}_0(k_m r)e^{\mp k_m z}\right]. \tag{33}$$

V. SERIES SOLUTION DERIVED FROM INTEGRAL SOLUTION

The series solution can also be derived from the integral solution of Sec. II. We start with Eq. (15) and first transform the solution to complex exponential form using the relation $2i\sin(kz)=e^{ikz}-e^{-ikz}$. Then

$$\phi_2(r,z) = -\frac{i\phi_b}{2\pi}\int_{-\infty}^{\infty} dk\frac{F_0(kr)}{kF_0(kb)}e^{ikz}. \tag{34}$$

We now follow through the effect of changing the sign of the original separation constant from k^2 to $-k^2$. Thus the $\pm k$ in the original solution becomes $\pm ik$. We choose $+ik$ for $z>0$ and $-ik$ for $z<0$ so that the z-dependent exponential in Eq. (34) does not diverge for large $|z|$.

$$\phi_2(r,z) = \mp \frac{i\phi_b}{2\pi}\int_{-i\infty}^{i\infty}\frac{F_0(\pm ikr)}{kF_0(\pm ikb)}e^{\mp kz}dk. \tag{35}$$

We can rewrite $F_0(\pm ikr)$ and $F_0(\pm ikb)$ by noting that $I_0(\pm ikr)=J_0(kr)$ and $K_0(\pm ikr)=(i\pi/2)[J_0(kr)+iY_0(kr)]$. Using Eq. (22) we obtain

$$\phi_2(r,z) = \mp \frac{i\phi_b}{2\pi}\int_{-i\infty}^{i\infty}\frac{\tilde{F}_0(kr)}{k\tilde{F}_0(kb)}e^{\mp kz}dk. \tag{36}$$

To evaluate the integral in Eq. (36), consider the closed contour integral

$$\oint \frac{\tilde{F}_0(kr)}{k\tilde{F}_0(kb)}e^{\mp kz}dk \tag{37}$$

around the path shown in Fig. 3. The integrand has simple poles at $k=0$ and at the zeros of the equation

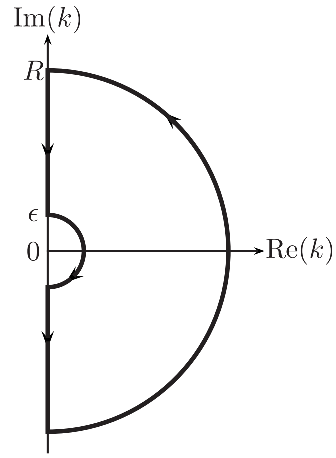

Fig. 3. Integration path for contour integral of Eq. (37).

$$\tilde{F}_0(kb) = J_0(kb)Y_0(ka) - J_0(ka)Y_0(kb) = 0. \tag{38}$$

Call the k values satisfying Eq. (38) k_m, where $m = 0,1,2,\ldots$. There are an infinite number of k_m, and they lie on the positive real k-axis. Although we have drawn the contour to avoid the pole at $k=0$, it will enclose all the k_m poles if we make ϵ small enough and let $R\to\infty$. For these conditions the residue theorem[21] gives

$$\oint \frac{\tilde{F}_0(kr)}{k\tilde{F}_0(kb)} e^{\mp kz} dk = 2\pi i \sum_{m=0}^{\infty} \lim_{k\to k_m}\left[(k-k_m)\frac{\tilde{F}_0(kr)}{k\tilde{F}_0(kb)} e^{\mp kz}\right]. \tag{39}$$

The evaluation of the right-hand side of Eq. (39) requires an application of L'Hospital's rule because both $k-k_m$ and $\tilde{F}_0(kb)$ are zero at $k=k_m$,

$$\lim_{k\to k_m} \frac{(k-k_m)}{\tilde{F}_0(kb)} = \left(\frac{d\tilde{F}_0(k_m b)}{dk}\right)^{-1}. \tag{40}$$

If we perform the derivative and use the relations $J_0'(z) = -J_1(z)$ and $Y_0'(z) = -Y_1(z)$, we obtain

$$\frac{d\tilde{F}_0(k_m b)}{dk} = a[J_1(k_m a)Y_0(k_m b) - J_0(k_m b)Y_1(k_m a)] + b[J_0(k_m a)Y_1(k_m b) - J_1(k_m b)Y_0(k_m a)] \tag{41a}$$

$$\equiv g(k_m, a, b), \tag{41b}$$

and Eq. (39) becomes

$$\oint \frac{\tilde{F}_0(kr)}{k\tilde{F}_0(kb)} e^{\mp kz} dk = 2\pi i \sum_{m=0}^{\infty} \frac{\tilde{F}_0(k_m r)}{k_m g(k_m, a, b)} e^{\mp k_m z}. \tag{42}$$

It remains to relate this contour integral to the original integral in Eq. (36). The integral is equal to the straight line portions of the contour integral when we take the limits $\epsilon\to 0$ and $R\to\infty$. We thus obtain the original integral by subtracting the contributions from the curved portions of the contour from the result of Eq. (42). The smaller curved portion of the contour can be evaluated by writing k in plane-polar form $k=\epsilon e^{i\theta}$ and integrating over the angle θ. Using the small-k limit for $\tilde{F}_0(kr)/\tilde{F}_0(kb)$ given in Eq. (16) and taking $\epsilon\to 0$ give $i\pi \ln(r/a)/\ln(b/a)$. Similarly, for the larger curved portion of the contour, we use $k=Re^{i\theta}$ and take the $R\to\infty$ limit. This integral gives zero.

We combine these contributions so that the series solution becomes

$$\phi_2(r,z) = \pm \phi_b \left[\frac{1}{2}\frac{\ln(r/a)}{\ln(b/a)} + \sum_{m=0}^{\infty} \frac{\tilde{F}_0(k_m r)}{k_m g(k_m, a, b)} e^{\mp k_m z}\right]. \tag{43}$$

Because we know that $\phi(r,0)=0$, we also obtain the following series identity:

$$-\sum_{m=0}^{\infty} \frac{\tilde{F}_0(k_m r)}{k_m g(k_m, a, b)} = \frac{1}{2}\frac{\ln(r/a)}{\ln(b/a)}. \tag{44}$$

By employing Eq. (38) and the Wronskian relation[15]

$$J_1(z)Y_0(z) - J_0(z)Y_1(z) = \frac{2}{\pi z}, \tag{45}$$

it can be shown that each term in the sum on the left-hand side of Eq. (44) is the same as in Eq. (32).

As before, we can compare the solution in Eq. (43) with our result in Eq. (15) and obtain an alternate form of the integral identity

$$\int_0^\infty \frac{F_0(kr)}{kF_0(kb)} \sin(kz) dk = \pm \pi \left[\frac{1}{2}\frac{\ln(r/a)}{\ln(b/a)} + \sum_{m=0}^{\infty} \frac{\tilde{F}_0(k_m r)}{k_m g(k_m, a, b)} e^{\mp k_m z}\right]. \tag{46}$$

VI. CONCLUSION

We have solved an electrostatics problem of practical interest using various approaches and a variety of mathematical techniques. For the asymmetric part of the problem, we have explored the consequences of the choice of the sign of the separation constant by obtaining both an integral and a series solution. The connection between the two solutions was made through the use of contour integration. The solutions also produce several integral and series identities, which we believe are new.

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APPENDIX A: ORTHOGONALITY RELATION

We apply tabulated integrals to obtain an orthogonality relation applicable to the direct series solution in Sec. IV. Integral 5.54.1 from Ref. 16 is

$$\int rR_p(\alpha r)S_p(\beta r)dr = \frac{\beta r R_p(\alpha r)S_{p-1}(\beta r) - \alpha r R_{p-1}(\alpha r)S_p(\beta r)}{\alpha^2 - \beta^2}, \tag{A1}$$

where R_p and S_p are solutions to Bessel's equation of order p

and $\alpha \neq \beta$. We can apply this integral to the function $\tilde{F}_0$ because it is a solution to Bessel's equation of order zero. By using two of the allowed k-values for α and β, we obtain

$$\int r\tilde{F}_0(k_m r)\tilde{F}_0(k_n r)dr = \frac{k_n r\tilde{F}_0(k_m r)\tilde{F}_{-1}(k_n r) - k_m r\tilde{F}_{-1}(k_m r)\tilde{F}_0(k_n r)}{k_m^2 - k_n^2}, \quad (A2)$$

where

$$\tilde{F}_{-1}(kr) = J_{-1}(kr)Y_0(ka) - J_0(ka)Y_{-1}(kr). \quad (A3)$$

If we make the limits of our integral a and b, the right-hand side of Eq. (A2) is zero because $\tilde{F}_0(k_m a)$ and $\tilde{F}_0(k_m b)$ are both zero for any k_m.

If the two k-values are the same, we employ integral 5.54.2 of Ref. 16 and obtain

$$\int r[\tilde{F}_0(k_m r)]^2 dr = \frac{r^2}{2}([\tilde{F}_0(k_m r)]^2 - \tilde{F}_{-1}(k_m r)\tilde{F}_1(k_m r)), \quad (A4)$$

where

$$\tilde{F}_1(kr) = J_1(kr)Y_0(ka) - J_0(ka)Y_1(kr). \quad (A5)$$

Again, for integral limits a to b, the first term on the right-hand side of Eq. (A4) vanishes. If we note that $J_{-1}(kr) = -J_1(kr)$ and $Y_{-1}(kr) = -Y_1(kr)$, we can write Eq. (A4) as

$$\int_a^b r[\tilde{F}_0(k_m r)]^2 dr = \frac{b^2[\tilde{F}_1(k_m b)]^2 - a^2[\tilde{F}_1(k_m a)]^2}{2}. \quad (A6)$$

If we combine Eq. (A6) with the result for unlike k-values, we have

$$\int_a^b r\tilde{F}_0(k_m r)\tilde{F}_0(k_n r)dr = \delta_{mn}\frac{b^2[\tilde{F}_1(k_m b)]^2 - a^2[\tilde{F}_1(k_m a)]^2}{2} \equiv \frac{\delta_{mn}}{2}h(k_m,a,b), \quad (A7)$$

where δ_{mn} is the Kronecker delta.

APPENDIX B: INTEGRAL EVALUATION

In Sec. 5.1 of Ref. 20, Watson proved the relation

$$\int z^{\nu+1}\left[B''(z) + \frac{2\nu+1}{z}B'(z) + B(z)\right]\mathcal{C}_\nu(z)dz = z^{\nu+1}[B'(z)\mathcal{C}_\nu(z) + B(z)\mathcal{C}_{\nu+1}(z)], \quad (B1)$$

where $B(z)$ is an arbitrary function and $\mathcal{C}_\nu(z)$ satisfies the recurrence relations

$$\mathcal{C}_{\nu-1}(z) + \mathcal{C}_{\nu+1}(z) = \frac{2\nu}{z}\mathcal{C}_\nu(z) \quad (B2)$$

and

$$\mathcal{C}_{\nu-1}(z) - \mathcal{C}_{\nu+1}(z) = 2\mathcal{C}'_\nu(z). \quad (B3)$$

Note that because both Bessel functions $J_\nu(z)$ and $Y_\nu(z)$ satisfy these relations, so do the functions $\tilde{F}_0$ and $\tilde{F}_1$. For $\nu = 0$ Eq. (B1) becomes

$$\int z\left[B''(z) + \frac{1}{z}B'(z) + B(z)\right]\mathcal{C}_0(z)dz = z[B'(z)\mathcal{C}_0(z) + B(z)\mathcal{C}_1(z)]. \quad (B4)$$

If we now choose $B(z) = \ln(\alpha z)$, Eq. (B4) reduces to

$$\int z\ \ln(\alpha z)\mathcal{C}_0(z)dz = C_0(z) + z\ \ln(\alpha z)\mathcal{C}_1(z). \quad (B5)$$

We now apply Eq. (B5) to our problem. We substitute $\tilde{F}$ for $\mathcal{C}$ and use $z = k_m r$ and obtain

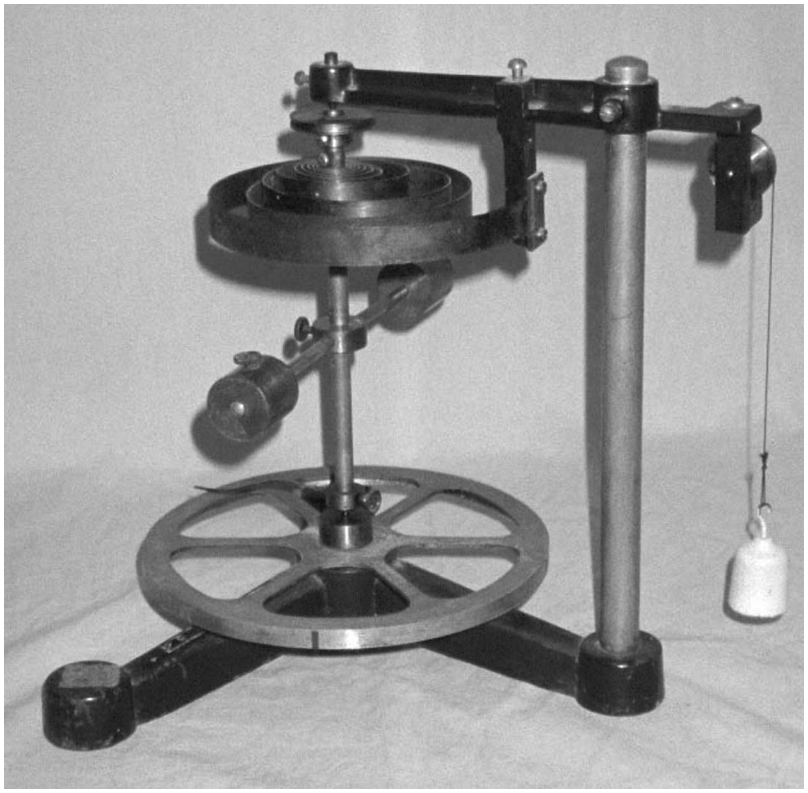
$$\int_a^b r\tilde{F}_0(k_m r)\ln\left(\frac{r}{a}\right)dr = \int_{k_m a}^{k_m b}\frac{z}{k_m}\tilde{F}_0(z)\ln\left(\frac{z}{k_m a}\right)\frac{dr}{k_m} = \frac{1}{k_m^2}\left[\tilde{F}_0(z) + z\ \ln\left(\frac{z}{k_m a}\right)\tilde{F}_1(z)\right]\Bigg|_{k_m a}^{k_m b}. \quad (B6)$$

Finally, we note that $\ln(1) = 0$ and F_0 vanishes at both limits, and we have

$$\int_a^b r\tilde{F}_0(k_m r)\ln(r/a)dr = \frac{b}{k_m}\ln(b/a)F_1(k_m b). \quad (B7)$$

[1] For a sample of current work, see the papers in *Non-Neutral Plasma Physics VII*, edited by J. R. Danielson and T. S. Petersen (American Institute of Physics, Melville, NY, 2009).
[2] John D. Jackson, *Classical Electrodynamics*, 2nd ed. (Wiley, New York, 1975), pp. 582–584.
[3] Francis F. Chen, *Introduction to Plasma Physics* (Plenum, New York, 1974), pp. 19–21.
[4] J. H. Malmberg and C. F. Driscoll, "Long-time containment of a pure electron plasma," Phys. Rev. Lett. **44**, 654–657 (1980).
[5] D. L. Eggleston, "Confinement of test particles in Malmberg–Penning trap with a biased axial wire," Phys. Plasmas **4**, 1196–1200 (1997).
[6] D. L. Eggleston, T. M. O'Neil, and J. H. Malmberg, "Collective enhancement of radial transport in a nonneutral plasma," Phys. Rev. Lett. **53**, 982–984 (1984).
[7] J. Notte and J. Fajans, "The effect of asymmetries on non-neutral plasma confinement time," Phys. Plasmas **1**, 1123–1127 (1994).
[8] J. M. Kriesel and C. F. Driscoll, "Two-regimes of asymmetry-induced transport in non-neutral plasmas," Phys. Rev. Lett. **85**, 2510–2513 (2000).
[9] D. L. Eggleston and B. Carrillo, "Frequency dependence of asymmetry-induced transport in a non-neutral plasma trap," Phys. Plasmas **10**, 1308–1314 (2003).
[10] Mary L. Boas, *Mathematical Methods in the Physical Sciences* (Wiley, New York, 1966), pp. 601–607.
[11] David J. Griffiths, *Introduction to Electrodynamics*, 3rd ed. (Prentice-Hall, Englewood Cliffs, NJ, 1999), pp. 369–370.
[12] Ronald N. Bracewell, *The Fourier Transform and Its Applications*, 2nd ed. (McGraw-Hill, New York, 1978), p. 394.
[13] Wolfram Research, MATHEMATICA, Version 6.0, ⟨www.wolfram.com/⟩.
[14] D. L. Eggleston and B. Carrillo, "Amplitude scaling of asymmetry-induced transport in a non-neutral plasma trap," Phys. Plasmas **9**, 786–790 (2002).
[15] Milton Abramowitz and Irene A. Stegun, *Handbook of Mathematical*

Functions with Formulas, Graphs, and Mathematical Tables (Dover, New York, 1972), pp. 358–389.
[16] I. S. Gradshteyn and I. M. Ryzhik, *Tables of Integrals, Series, and Products* (Academic, New York, 1965), p. 405.
[17] Samuel M. Selby, *CRC Standard Mathematical Tables* (Chemical Rubber, Cleveland, 1969), p. 455.
[18] William R. Smythe, *Static and Dynamic Electricity*, 3rd ed. (McGraw-Hill, New York, 1968), pp. 179–190.
[19] Reference 2, pp. 102–110.
[20] G. N. Watson, *A Treatise on the Theory of Bessel Functions* (Cambridge U. P., Cambridge, 1922), pp. 132–133.
[21] Murray R. Spiegel, *Complex Variables with an Introduction to Conformal Mapping and Its Applications* (McGraw-Hill, New York, 1964), pp. 179–184.



Torsional Oscillator. This torsional oscillator in the demonstration collection of the University of Texas at Austin, was made by Gurley of Troy, New York, best known for its surveying instruments. The vertical shaft is pivoted, and the masses on the sliding cross-rod can be slid in and out to vary the moment of inertia of the oscillating system. The spiral spring provides linear restoring torque in both angular directions. The torsion constant of the spring is determined by hanging various masses on one end of the string; the other end is wrapped around a pulley of known radius. The period of the system can be measured as a function of the moment of inertia of the system: the square of the period is proportional to the moment of inertia. (Photograph and Notes by Thomas B. Greenslade, Jr., Kenyon College)