1. Stability of geodesic wave maps

As a part of my Ph.D. thesis I studied the stability properties of a special class of solutions to the wave maps system. The wave maps, or harmonic maps of Minkowski space, are maps from the Minkowski space $\mathbb{R}^{1+n}$ into a Riemannian manifold $M$, which are stationary for the Lagrangian

$$\int_{\mathbb{R}^{1+n}} g_{\alpha\beta}(\partial_{\alpha}\phi, \partial_{\beta}\phi) dx dt,$$

where $(g_{\alpha\beta}) = \text{diag}\{-1, 1, ... , 1\}$ is the standard Minkowski metric, and $h$ is the positive definite metric of the $m$ dimensional Riemannian manifold $M$. As solutions to the time-dependent variational problem, wave maps generalize both the geodesic curves $\gamma : \mathbb{R} \to M$, and the harmonic maps $\varphi : \mathbb{R}^n \to M$. In the intrinsic form, the Euler-Lagrange equations for the Lagrangian can be written as

$$D^{\alpha}\partial_{\alpha}\phi = 0,$$

where $D^{\alpha}$ is the covariant derivative, and we use the Minkowski metric $g$ to raise and lower the indices. This equation, which in the special case of $M = S^n$ is called the nonlinear $\sigma$-model, also arises from reductions of Yang-Mills system and Einstein’s field equations under specific symmetry conditions. The equation allows a special class of solutions, the \textit{geodesic wave maps} $\phi(t,x) = \gamma(u(t,x))$, where $\gamma : \mathbb{R} \to M$ is a geodesic curve, and $u : \mathbb{R}^{1+n} \to \mathbb{R}$ is a free wave, \textit{i.e.} it solves the linear wave equation $\Box u = 0$.

The wave maps system is invariant with respect to the scaling $\phi(t,x) \mapsto \phi(\lambda t, \lambda x)$, and the Cauchy problem

$$\begin{cases}
D^{\alpha}\partial_{\alpha}\phi = 0, \\
(\phi(0,x), \partial_{t}\phi(0,x)) = (\phi_0, \phi_1) : \mathbb{R}^n \to TM,
\end{cases}$$

is called $H^s$ critical, if the homogeneous Sobolev norm $\tilde{H}^s$ of the initial data is invariant under the above scaling. In our case the critical exponent is $s_c = \frac{n}{2}$, and as a general principle, one usually expects local well-posedness for subcritical exponents $s > s_c$, and global well-posedness for the critical exponent $s_c$, provided the initial data is small in the critical Sobolev space $H^{s_c}$. For exponents $s < s_c$ some form of ill-posedness is expected. The local well-posedness for the wave maps system in subcritical Sobolev spaces was established by Klainerman-Machedon \cite{28, 29} (in dimensions $n \geq 3$), and Klainerman-Selberg \cite{32} (in dimension $n = 2$; see \cite{33} for a unified approach to all dimensions $n \geq 2$) (see also \cite{70}). The global well-posedness for small data at critical regularity was first shown by Tataru \cite{62, 64}, who considered the Cauchy problem in the critical Besov space $\dot{B}_2^{2,1}$ instead of the critical Sobolev space. In the Besov space approach one has the advantage of having a control of the $L^\infty$ norm of the solution, while the $\tilde{H}^\frac{n}{2}$ norm barely fails to do so. This allowed Tataru to neglect the geometry of the target manifold, and using the finite speed of propagation, to only study the system in local coordinates,

$$\Box v^i + \Gamma^i_{j,k}(v)Q_0(v^j, v^k) = 0,$$

where $\Gamma^i_{j,k}$ are the Cristoffel symbols of the target manifold, $Q_0(f,g) = \partial_\alpha f \partial^\alpha g$, and $v = (v^i) : \mathbb{R}^{1+n} \to \mathbb{R}^m$ is the vector of the components of the wave map in the local coordinates. Tataru’s breakthrough was followed by Tao’s seminal results \cite{54, 55} establishing global regularity for wave
maps into the sphere for small data in the critical Sobolev spaces (see also [31, 35, 36, 44, 46] for generalizations to more general targets).

The question of stability of geodesic wave maps was first considered by Sideris [47], who used the commuting vector fields approach to show that in dimension \( n = 3 \), if one starts with an initial data close to that of the geodesic wave map in the sense of weighted Sobolev type norms with regularity \( s \geq 10 \), then there exists a global solution continuing the initial data, and it remains close to the geodesic wave map in the same norms. In my thesis I used nonlinear Fourier analysis techniques to relax the assumption of smallness, requiring the data to be small only in the sense of a subcritical Sobolev norm with regularity \( s = \frac{n}{2} + \epsilon \). Similar to Sideris, we consider perturbations of the geodesic wave map in the Fermi chart in a neighborhood of \( \gamma \) in \( M \). The Cauchy problem for the perturbed wave map can then be written in the form

\[
\left\{ \begin{align*}
\square V + \Gamma(V)Q_0(V, V) + \Gamma(V)Q_0(V, u) + \Gamma(V)Q_0(u, u) &= 0, \\
(V, \partial_t V)|_{t=0} &= (V_0, V_1),
\end{align*} \right. \tag{WM}
\]

where the first component of \( V \) measures the perturbation in the direction along the geodesic, and the coefficients \( \Gamma \in C^\infty(\mathbb{R}^m) \), that further satisfy \( \Gamma(0) = 0 \). Our main result is the following.

**Theorem** ([13, 14]). Let \( u_0, u_1 \in C^\infty_c \), and \( u \) is the solution to the Cauchy problem \( \square u = 0 \); \( (u, \partial_t u)|_{t=0} = (u_0, u_1) \). Let \( n \geq 3 \), and \( s > \frac{n}{2} \). There exists a positive small constant \( \epsilon > 0 \), such that if \( \| (V_0, V_1) \|_{H^{s'} \times H^{s'-1}} < \epsilon \), then, there exists a global solution \( V : \mathbb{R}^{1+n} \to \mathbb{R}^m \) to (WM), satisfying \( \| (V, \partial_t V) \|_{L^\infty_t(\mathbb{R}, B^{s-1}_{n2} \times B^{s'-1}_{n2})} \leq C \epsilon \), where \( C = C(u, s) \), i.e. is independent of \( \epsilon \).

The perturbed map remains close to the geodesic wave map also in the homogeneous Sobolev spaces \( H^{s'} \) for the exponents \( \frac{n}{2} \leq s' \leq s \), i.e., \( V[t] = (V, \partial_t V)(t) \) remains small in the spaces \( H^{s'} \times \dot{H}^{s'-1} \) for all time. Our equations for \( V \) differ from those of the wave map system in local coordinates by an extra nonlinear term, which is essentially only linear in \( V \), and quadratic in the free wave \( u \), which are large in norm. To get around the largeness issue, we employed a method reminiscent of Krieger’s [37] approach to establishing stability of spherically symmetric 2D wave maps, in which one continues the local solution to a large time, by possibly shrinking the initial data, after which the dispersion of the linear operator kicks in, allowing control of the extra term. In contrast to Krieger’s stability result, where the target is the hyperbolic space \( \mathbb{H}^2 \), and the closeness is obtained for the derivatives of the maps in the Coulomb gauge, which itself changes with the perturbed map, our result shows the closeness of the maps themselves.

Since this work was completed, Tao [56, 57, 58, 59, 60, 61], Sterbenz-Tataru, [52, 53] and Krieger-Schlag [38] announced results establishing global regularity of 2D finite energy wave maps. Being large data results, these also establish regularity for wave maps with initial data close to those of geodesic wave maps in dimension \( n = 2 \).

## 2. The 2D quadratic derivative NLW.

Consider the Cauchy problem for the following nonlinear wave equation with quadratic derivative nonlinearity in \( \mathbb{R} \times \mathbb{R}^n \),

\[
\left\{ \begin{align*}
\square u^f &= \partial u^f \partial u^K, \\
(u, u_t)|_{t=0} &= (f, g),
\end{align*} \right. \tag{NLW}
\]

where \( u = (u^f) : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^m \). A particular example of (NLW) is the following Cauchy problem, in which the quadratic derivative nonlinearity has a special algebraic structure.

\[
\left\{ \begin{align*}
\square u^f &= Q(u^f, u^K), \\
(u, u_t)|_{t=0} &= (f, g),
\end{align*} \right. \tag{Q-NLW}
\]
where the nonlinear term $Q$ is one of the following null/forms:

$$Q_{ij}(v, w) = \partial_i v \partial_j w - \partial_j v \partial_i w, \quad \text{or} \quad Q_{0j}(v, w) = \partial_t v \partial_j w - \partial_j v \partial_t w.$$ 

These forms, along with the form $Q_0(v, w) = \partial_t v \partial_t w - \nabla v \cdot \nabla w$, are the three basic null forms of Klainerman [26] (see also [27, 8]). The equation (Q-NLW) serves as a model for the space-time Monopole equation (ME) in the Coulomb gauge (with $Q = Q_{ij}$ as a part of its nonlinearity) and the Ward Wave map (WWM) equation (with $Q = Q_{0j}$ as a part of its nonlinearity), both of which were introduced by Ward in [68] and [67] respectively (see [11] for a broad survey on ME and the connection with WWM). The Monopole equation arises as a dimensional reduction of the self-dual Yang-Mills equation with split signature $R^{2,2}$, while Ward wave maps equation can be realized as a dimensional reduction of the anti-self-dual Yang-Mills equation. Both equations are examples of completely integrable systems in $R^{1+2}$, and have Lax pair formulations. The appearance of null-form structure in these equations was unveiled and exploited by Czubak in [9, 10], to establish local well-posedness for the Ward wave map and the space-time Monopole equations in the Coulomb gauge.

The null-forms $Q_0, Q_{ij}, Q_{0j}$ were first encountered in the study of global existence for small smooth data of physical and geometric equations for which (Q-NLW) serves as a model. Further work focused on understanding the global existence theory for physical wave equations with null-form structure in their nonlinearities and possessing a natural conserved energy, assuming this to be bounded or small. Since the energy lives at the level of $H^1$ regularity, such existence theories need to rely on machinery developed in the setting of low regularity Sobolev spaces. Note that equation (NLW), and hence, also (Q-NLW), is invariant under the scaling $u_\lambda(t, x) = u(\lambda t, \lambda x)$, and hence is critical relative to scaling for data in $\dot{H}^{\frac{n}{2}} \times \dot{H}^{\frac{n}{2} - 1}$. From scaling considerations one might expect local well-posedness for initial data in the spaces $H^s \times H^{s-1}$ with $s > \frac{n}{2}$ (subcritical regime). However, counterexamples of Lindblad [39], [40], [41] further restrict the regularity assumption on initial data for local well-posedness. In view of these counterexamples, one can only expect local well-posedness for (NLW) for

$$s > \max \left\{ \frac{n}{2}, \frac{n+5}{4} \right\}. \quad (1)$$

For dimensions $n \geq 5$ this range coincides with the subcritical range $s > \frac{n}{2}$, for which local well-posedness was proved by Tataru [63] using refinements of $X^{s,b}$ spaces.

In dimensions $n = 4$ one can show local well-posedness for the range $s > \frac{9}{4}$, which is sharp by (1), using the regular $X^{s,b}$ spaces (see [12] for related bilinear estimates).

In dimensions $n = 3$, Ponce and Sideris [45] proved LWP for $s > 2$ using Strichartz estimates approach. This is again sharp in light of the counterexamples of Linblad. If the nonlinearity has a null-form structure, Klainerman and Machedon [30] improved the range for LWP to $s > \frac{3}{2}$, which is almost critical in 3D.

In dimension $n = 2$, the sharp LWP result for $s > \frac{7}{4}$ can be again shown using the Strichartz estimates approach. If the nonlinearity has a null-form structure, i.e. for (Q-NLW), the LWP results can again be improved. For the $Q_0$ null-form, almost critical LWP was established by Klainerman and Selberg [32] in the context of wave maps. For the other null forms, Zhou [69] established LWP for data in $H^s$ with $s > \frac{5}{4}$, and showed that this is sharp.

So contrary to the expectations from the scaling analysis, one does not reach the critical regularity for the local well-posedness for either (NLW) or (Q-NLW). This obstruction for the null-form problem also forced a corresponding gap of $\frac{1}{4}$ derivative above the scaling regularity in the local well-posedness of the (ME) and (WMM) equations [9, 10].
An alternative approach to low regularity well-posedness problem is to consider data in spaces other than the Sobolev spaces, but which still have similar scaling properties. One candidate for such spaces is the family of Fourier-Lebesgue spaces $\hat{H}^r_s$, defined by the norm

$$\|u\|_{\hat{H}^r_s} = \|\langle \xi \rangle^s \hat{u}\|_{L^r_{t,x}},$$

where $r'$ is the Lebesgue conjugate of $r$. These spaces have proved fundamental in lowering the regularity and closing such gaps for a variety of equations (see e.g. [21, 6, 66, 17, 18, 7, 20, 19]), where ill-posedness of some form occurs above the scaling regularity for data in Sobolev spaces. In the context of the wave equation, Grünrock [19] recently obtained almost optimal local well-posedness for (NLW) in $\hat{H}^r_s$ spaces in dimension $n = 3$. Observe that in $\mathbb{R}^n$ the critical exponent for our equations for data in $\hat{H}^r_s$ is $s_c = \frac{n}{2}$, and in general, the space $\hat{H}^r_s$ scales like the Sobolev space $H^\sigma$, if $\sigma = s + n \left(\frac{1}{2} - \frac{1}{r}\right)$. In [19] Grünrock established local well-posedness in three dimensions for local data in $\hat{H}^r_s$, provided $1 < r \leq 2$ and $s > 2/r + 1$. For $r = 2$, $\hat{H}^r_s = H^s$, and his result coincides with the earlier result of Ponce-Sideris [45], which is sharp in the Sobolev scale in view of the counterexamples of Lindblad. However, the other endpoint, $(s, r) = (3, 1)$, is almost attained by Grünrock’s result, it is critical for Fourier-Lebesgue spaces, and has the same scaling as the critical Sobolev space $H^\frac{3}{2}$ in 3D. Thus, his work, in some sense, circumvents the counterexamples of Lindblad, which had left a gap of $\frac{1}{2}$ derivative on the Sobolev scale, thereby closing such a gap in the $\hat{H}^r_s$ spaces. In his paper Grünrock relies on a calculation of Foschi-Klainerman [12], which are particularly advantageous in $n = 3$ dimensions, while introducing uncooperative multipliers that blow up along the null-cone $|\tau| = |\xi|$ in $n = 2$ dimensions, preventing a direct extension of his result to our 2-dimensional problem (NLW). However, the null-forms $Q_{0j}$ and $Q_{ij}$ have enough cancellations along the null-cone to neutralize this multiplier, which allowed us to prove necessary bilinear estimates for the null-forms in $X_{s,b}^r$ spaces, which are the analogs of $X^{s,b}$ spaces in the context of Fourier-Lebesgue spaces. These estimates coupled with a standard local well-posedness scheme allowed us to establish the following result.

**Theorem** ([15]). The Cauchy problem ($Q$-NLW) is locally well-posed in dimension $n = 2$ for $(f, g) \in \hat{H}^r_s \times \hat{H}^r_{s-1}$, provided $1 < r \leq 2$, $s > \frac{3}{2r} + \frac{1}{2}$.

Note that one endpoint of our range of exponents is $(s, r) = (\frac{5}{4}, 2)$, which exactly coincides with the sharp Sobolev result for data in $H^{\frac{2}{3}+}$, while the other endpoint, $(s, r) = (2, 1)$, is critical in dimension $n = 2$. Thus, our result is almost critical for $\hat{H}^r_s$ spaces, closing the $\frac{1}{4}$ derivative gap left for data in Sobolev spaces.

We also prove appropriate multiplicative properties of the solution space $X_{s,b}^r$, which allows us to close the estimates for the cubic nonlinearities appearing in the Ward wave map equations, which can be written as

$$\Box J + JQ_0(J^{-1}, J) + JQ_{02}(J^{-1}, J) = 0,$$

(WWM)

where $U : \mathbb{R}^{1+2} \to U(2)$ is a unitary matrix valued function. Thus, we also obtain almost critical well-posedness of the Ward maps in the Lebesgue-Fourier spaces $\hat{H}^r_s$.

Since the appearance of our manuscript in the preprint arXive, Tesfahun [65] relied on some of our calculations to show almost critical well-posedness for the Monopole equation in Lorenz gauge with data in Fourier-Lebesgue spaces. We believe that our ideas will be similarly instrumental in closing the gap to the almost criticality in the local well-posedness theory for other geometric and physical field equations that exhibit $Q_{\mu\nu}$ null-forms, such as the Maxwell-Klein-Gordon and Chern-Simons-Higgs equations. These questions are currently under investigation.
Problem 1. Adapt the null-form estimates of [15] to the context of Monopole and Maxwell-Klein-Gordon equations in the Coulomb gauge, towards establishing almost optimal local well-posedness for these problems.

This is work in progress with Magdalena Czubak (Un. of Binghamton) and Andrea Nahmod (UMass Amherst).

As our result [15] establishes almost critical well-posedness for the Ward wave map in two dimensions, we believe that the question of global regularity with small data in the critical Fourier-Lebesgue spaces should now be within reach.

Problem 2. Establish global regularity for the Ward wave map problem (WWM) with small data at the critical Fourier-Lebesgue regularity.

Our line of attack for this problem is to adapt the solution spaces of Tataru [64] and/or Tao [55] to the framework of Fourier-Lebesgue spaces and use the bootstrapping method of Tao with the frequency envelopes to obtain a uniform in time control on a subcritical norm of the solution, thus leading to a global in time result. This approach requires careful analysis of all the possible frequency interactions in the nonlinearity, and a renormalization approach to gauge away the worst interactions.

As the approach used in establishing the above result for (Q-NLW) is not useful for the general quadratic derivative nonlinearity problem (NSW), we are left with the task of establishing bilinear estimates in $X_{r,b}^s$ spaces without relying on Foschi-Klainerman calculations for free waves. To this end, we decompose the inputs into their dyadic frequency pieces, and consider all possible frequency interactions separately. For $r = 2$ all almost sharp such estimates were established by D’Ancona, Foschi and Selberg [3] (dimensions $n \geq 3$), [4] (dimensions $n = 1, 2$). We extend some of their estimates to $X_{r,b}^s$ spaces in 2D, which allowed us to establish the following result for (NLW).

**Theorem** ([16]). The Cauchy problem (NLW) is locally well-posed in dimension $n = 2$ for $(f, g) \in \hat{H}_r^s \times \hat{H}_r^{s-1},$ provided $\frac{3}{2} < r \leq 2, \ s > \frac{3}{2r} + 1.$

For $r = 2$ this result coincides with the sharp Sobolev space result, while giving improvement for the other values of $r$ in the allowed range. The other endpoint, $(s,r) = (2, \frac{3}{2})$ corresponds to the Sobolev exponent $\frac{5}{3} = \frac{7}{4} - \frac{1}{12},$ and thus, can be thought of as a $\frac{1}{12}$ derivative improvement on the sharp Sobolev result. Unfortunately, our estimates do not extend to the full range $1 < r \leq 2$ due to a diverging term which is forced upon us by the loss of self-duality of $L^r$ when $r \neq 2$. We are currently investigating alternative ways of deriving necessary bilinear estimates, which would avoid such divergences, and possibly allow an improvement for the range of $r$.

Another approach for the local well-posedness for (NLW) is to consider randomized data and exploit the nonlinear smoothing that this random data induces. As was observed by Burq and Tzvetkov in their seminal paper [5], the randomization allows one to obtain better summability properties (an expression of smoothing, though not in terms of derivatives) for Strichartz type estimates.

Problem 3. Study the local well posedness of the Cauchy problem (NLW) in two dimensions for randomized data, exploiting the improved summability properties arising from randomization.

Incidentally, Burq-Tzvetkov’s data randomization mechanism may also be implemented within the framework of Fourier Lebesgue spaces, where one has a direct connection between summability and regularity.

This is work in progress with Andrea Nahmod and Luc Rey-Bellet (UMass Amherst).
3. Incompressible elasticity in exterior domains.

Consider deformations of a homogeneous elastic material filling the space, given by the deformation vector \( \eta : \mathbb{R} \times \mathbb{R}^3 \to \mathbb{R}^3 \), which is an orientation preserving diffeomorphism mapping the material point \( X \in \mathbb{R}^3 \) to its position \( \eta(t, X) \in \mathbb{R}^3 \) at time \( t \). The deformation gradient \( F_t^i = D_t \eta^i \) then satisfies the condition \( J(\eta) = \det(F) > 0 \). The equations of motion governing the deformation arise from the principle of stationary action applied to the Lagrangian

\[
\mathcal{L}[\eta] = \int \int \left[ \frac{1}{2} |D_t \eta|^2 - W(D\eta) \right] \, dx \, dt.
\]

Here \( W(D\eta) \) is the strain energy density function characterizing the hyperelastic material. The Euler-Lagrange equations for compressible elasticity are then

\[
D_t^2 \eta^i - D_t \left( \frac{\partial W}{\partial F_t^i}(D\eta) \right) = 0.
\]

Considering deformations in the form \( \eta(t, X) = X + u(t, X) \) for small displacements \( u(t, X) \) from the reference configuration, we can write the equations for the displacement, which in the case of isotropic materials take the form

\[
D_t^2 u^i - c_2^2 \Delta u + (c_1^2 - c_2^2) D_t D \cdot u + B_{lmn}^i D_n u^k D_l D_m w^j + \cdots = 0. \tag{CE}
\]

The dots above represent terms that are at least of third order, and the positive constants \( c_1, c_2 \) are the propagation speeds of pressure and shear waves respectively. The higher order terms do not affect the long-time behavior of small elastic displacements, and it is customary to truncate the equations at the quadratic level.

The almost global existence for the compressible isotropic hyperelasticity (CE) was established by John [22]. He relied on the commuting vector field approach, making use of the classical invariance of the linear elasticity operator, as well as estimates for the fundamental solution of the associated linear equation. Klainerman and Sideris [34] established the same result using only classical invariance and Sobolev type decay estimates, thus bypassing estimation of the fundamental solution of the linear operator. The life-span bounds in these results are sharp in view of John’s counterexamples [23]. However, imposing an additional null-from structure in the quadratic nonlinearities for the pressure waves allowed Sideris [48, 49] to establish global existence for the compressible elastodynamics (see also [1, 2]).

For incompressible materials one has the constraint \( J(\eta) = \det(D\eta) \equiv 1 \) and the equation for the displacement in the Euclidean space takes the form

\[
D_t^2 u^i - c_2^2 \Delta u + (c_1^2 - c_2^2) D_t D \cdot u + B_{lmn}^i D_n u^k D_l D_m w^j + \cdots = \nabla_\eta p \quad \tag{iCE}
\]

where \( p \) is the pressure, and \( \nabla_\eta p \) is its Eulerian gradient, i.e. \( \nabla_\eta p = (\nabla(p \circ \eta^{-1})) \circ \eta \). Incompressibility also implies the constraint \( \nabla_\eta(D_t u) = 0 \) on the velocity. Sideris and Thomases established the global existence for the incompressible elasticity in the Euclidean space first via the isotropic limit [50], and then directly [51]. Due to the constraints being naturally written in the Eulerian coordinates, they found it useful to study the first order system in Eulerian coordinates,

\[
\begin{aligned}
\partial_t H_i^j + v \cdot \nabla H_i^j + H_j^j \partial_t v^i = 0, \\
\partial_t v^i + v \cdot \nabla v^i + \tilde{A}_{lm}^i(H) H_l^p \partial_t H_m^j + \partial_t p = 0,
\end{aligned}
\]

along with the constraints

\[
\begin{aligned}
\partial_t H_m^i = \partial_m H_i^1, \\
\det H = 1, \\
\nabla \cdot v = 0.
\end{aligned}
\]
Here $H = \nabla X = F^{-1}$, $v = D_t u$, and the linear part of $\dot{A}$ has the form $\dot{A}^{lm}_{ij} = \left( c_1^2 - c_2^2 \right) \delta^l_i \delta^m_j + c_2^2 \delta^{lm} \delta_{ij}$. To establish the global existence Sideris-Thomases made no restriction on the quadratic nonlinearities, since the null-condition is inherently satisfied for the shear waves, while the pressure waves vanish in the incompressible limit.

On the other hand, following the approach of Keel, Smith and Sogge for semilinear [24] and quasilinear [25] wave equations in the exterior domain, Metcalfe [42] established almost global existence for compressible elastodynamics exterior to star-shaped obstacles. Following this, global existence under a null-structure assumption was shown by Metcalfe and Thomases [43]. The key to the Keel-Smith-Sogge (KSS) approach is establishing $L^2_t L^2_x$ estimates that allow one to close the generalized energy estimates without relying on time decay of the solutions of the wave equation. This allows one to use only the generators of translations and spatial rotations, which essentially preserve the Dirichlet conditions, thus bypassing the use of the scaling vector field or the Lorentz boosts, which are problematic for the obstacle problem.

An interesting question is then whether one can adopt Metcalfe’s KSS approach to the case of incompressible elasticity in the exterior domain.

**Problem 4.** Establish global existence for incompressible elastic waves in the exterior domain via appropriate KSS estimates.

To use an iteration scheme based on Metcalfe-Sogge’s KSS estimate for the variable coefficient wave equation, one needs to estimate $\nabla \eta p$ in (ICE) in terms of the solution $u$. To this end the method of Sideris and Thomases of estimating this term for the incompressible elasticity in the Euclidean space seems promising. We also observe that unlike Metcalfe’s result for compressible elasticity in the exterior domain, in our case we expect global existence, due to the already mentioned inherent null-form structure for shear waves.

This is work in progress with John Helms and Thomas Sideris (UC Santa Barbara).

**References**


