Large-Amplitude Periodic Oscillations in Suspension Bridges

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Introduction to Suspension Bridges

Beginning Model of a Suspension Bridge

Partial Differential Equations Model

Traveling Waves on the Golden Gate Bridge

Considering Suspension Cables
Introduction to Suspension Bridges

- Above the roadway
  - Towers help keep the bridge upright
  - Main cables run concave up across the bridge
  - Suspender cables connect the main cable to the roadway

- Below the roadway
  - Large anchors act as counterweights
  - Piers cement towers to the ocean floor

Figure: Basic Diagram of a Suspension Bridge
Introduction to Suspension Bridges

- Suspension bridges are important structures in today’s world
- Akashi Kaikyo Bridge (spans a distance of 12,828 feet), Golden Gate Bridge, Tacoma Narrow Bridge
- Suspension bridges are light and flexible - this causes them to move and vibrate with the presence of wind or other external forces
Introduction to Suspension Bridges

- Many forces act on a suspension bridge
  - Compression applies balanced force to the pillars of the bridge
  - Tension is the pulling force that acts on the cables of the bridge
  - Forces act downward, and are distributed throughout the bridge from the tower to the cables and then suspenders and into the anchors
  - Forces acting on the cables by the deck and the weight of vehicles result in a parabola shape for main cables

- Suspension bridges have a live load, dead load, and dynamic load
  - Live loads consist of vehicles, wind, and changes in temperature and rain
  - Dead loads consist of the weight of all of the components of the bridge
  - Dynamic load refers to environmental factors such as earthquakes and wind storms

- Due to gravity and the weight of the suspension bridge it has a tendency to collapse
Mathematical Model

- We will begin by introducing a mathematical model for the behavior of the suspension bridge:

\[ u_{tt} + E l u_{xxxx} + \delta u_t = -k u^+ + W(x) + \epsilon f(x, t) \]  

(5.1)

with boundary conditions \( u(0, t) = u(L, T) = u_{xx}(0, t) = u_{xx}(L, t) = 0 \)

- Note that we are assuming a model for the suspension bridge with initial conditions equivalent to 0
- \( u(x, t) \) measures the downward deflection of the beam
- \( u_{tt} \) term is kinetic energy (PDE wave equation)
- \( E l u_{xxxx} \) term comes from the vibrating beam equation (forces on bridge)
- \( E \) represents the elastic modulus and \( l \) represents the area moment of inertia, therefore \( E l \) is overall stiffness
- \( \delta u_t \) is a dampening term
- \( k \) is the spring constant
- \( u^+ \) denotes \( u \) when \( u \) is positive and 0 everywhere else
- \( W(x) \) term represents the weight per unit length of the bridge
- \( \epsilon f(x, t) \) is external forcing term
Simplifying to an ODE

Lazer and McKenna used separation of variables and simplified the separated solution to only include the first Fourier mode. They replaced:

\[ W(x) \rightarrow W(x) = W_0 \sin\left(\frac{\pi x}{L}\right), \]

\[ f(x, t) \rightarrow f(x, t) = f(t) \sin\left(\frac{\pi x}{L}\right), \]

\[ u(x, t) \rightarrow u(x, t) = y(t) \sin\left(\frac{\pi x}{L}\right). \]

This results in the simplified model of:

\[ y'' + \delta y' + EI\left(\frac{\pi}{L}\right)^4 y + ky^+ = W_0 + \epsilon f(t) \quad (6.1) \]
Beginning Model of a Suspension Bridge

Periodic Solutions

- Now, using the ordinary differential equations model, Lazer and McKenna considered the periodic solutions of the form:

\[
y'' + f(y) = c + g(t)
\]

\[
y(0) = y(2\pi)
\]

\[
y'(0) = y'(2\pi)
\]

with \( f'(+\infty) = b \) and \( f'(-\infty) = a \)

- Assume that \( c \) is a constant that is a multiple of the first eigenfunction, and that \( g(t) \) is a periodic function.
- As the distance between \( a \) and \( b \) increases, then we will have that more and more oscillatory solutions exist, and their order of magnitude is that of \( c \).
- This means that the ratio between the magnitudes of the solutions will be approximately \( c \).
Periodic Solutions

- This implies that the larger the difference between $a$ and $b$ and the larger the intervals that the key parameter values lie in, the larger the magnitude of the oscillatory solutions.

- This can help explain why a suspension bridge could exhibit large wave behavior at an unsafe and unsustainable level.

- In this case the bridge has to rely more on the spring constant $k$ and less on the rigidity of the deck, which helps contribute to large magnitude waves traveling through the bridge.
Partial Differential Equations Model

\[ u_{tt} + u_{xxxx} + \delta u_t = -k u^+ + W_0 + \epsilon f(x, t) \]  \hspace{1cm} (9.1)

with boundary conditions

\[ u(0, t) = u(L, t) = u_{xx}(0, t) = u_{xx}(L, t) = 0. \]

- \( EI = 1 \) and \( W(x) = W_0 \) which is a constant
- The weight density is constant throughout the bridge

**Theorem 1**

Let \( \delta = 0, \ L = \pi \) and \( T = \pi \). In addition, let \( f(x, t) \) be an even function in time, \( T \)-periodic in time and even in \( x \) about \( \pi/2 \). Then if \( 0 < k < 3 \) then (9.1) has a unique periodic solution of period \( \pi \). If \( 3 < k < 15 \), then the equation has a large amplitude periodic solution.
Partial Differential Equations Model

- Over strengthening the bridge may increase the spring constant
- Too much tension does not allow the bridge to oscillate
- Set \( \delta = 0.01, \ E = 1, \ k = 18, \ W = 10 \) and the forcing term as \( \lambda \sin \mu t \sin \left( \frac{n\pi x}{L} \right) \). Various initial conditions of small or large amplitude were used.

- With a short bridge, \( L = 3 \), the solution would converge to different periodic solutions in finite time, over a large range of \( \lambda \) and \( \mu \)
- A longer bridge, say \( L = 6 \), becomes unstable and symmetry-breaking occurs.
- The solution converges to what appears to be a wave, traveling up and down the bridge, and being reflected at the end-points
Figure: Multiple solutions of PDE model with constant values and a forcing term as described above. Also with $n = 1$ and $\mu$ and $\lambda$ given.
One specific case that Lazer and McKenna took a look at was the nonlinear phenomena of traveling waves on the Golden Gate Bridge during a violent storm on February 9, 1938. The chief engineer of the bridge reported seeing the “suspended structure of the bridge was undulating vertically in a wavelike motion of considerable amplitude... the oscillations and deflections of the bridge were so pronounced they would seem unbelievable.”
Modelling Traveling Waves

To model solutions of this nature, and to show that their model allowed for traveling waves such as those observed on the Golden Gate Bridge, Lazer and McKenna started with PDE model

Assume there is no external forcing term ($\epsilon f(x, t) = 0$) and no damping term ($\delta = 0$)

Assume $k = W = 1$

Thus we now have the equation:

$$u_{tt} + u_{xxxx} = -ku^+ + W_0$$  \hspace{1cm} (13.1)

with boundary conditions $u(0, t) = u(L, t) = u_{xx}(0, t) = u_{xx}(L, t) = 0$. 
Traveling Waves

- Note that we have $u \equiv 1$ is an equilibrium solution.
  $$u_{tt} + u_{xxxx} = -k u^+ + W_0 \text{ becomes } 0 = -(1)(1) + 1 = -1 + 1 = 0.$$  

- To simplify to an ODE model, Lazer and McKenna looked for solutions of the form $u(x, t) = 1 + y(x - ct)$.

- Thus we have that $y$ depends on both $x$ and $t$ and satisfies the equation:
  $$y^{'''} + c^2 y'' + (y + 1)^+ = 1 \quad (14.1)$$

  where $y(+\infty) = y(-\infty) = 0$
Lazer and McKenna solved this ODE by solving a system of linear ODEs and the solution can be seen in the Figure below. This Figure is a model of a traveling wave similar to those that was seen at the Golden Gate Bridge.

Figure : Traveling Wave Solution of (14.1)
Consider is the motion of the cables in a suspension bridge. The cable will be treated as a vibrating string coupled with the vibrating beam of the roadbed. The roadbed will act as nonlinear springs with a spring constant given by $k$. There will be no restoring force if the springs are compressed. The model for this system is given by:

\[
\begin{align*}
    m_1 v_{tt} - T v_{xx} + \delta_1 v_t - k(u - v)^+ &= \epsilon f_1(x, t) \\
    m_2 u_{tt} + E I u_{xxxx} + \delta_2 u_t + k(u - v)^+ &= W_1
\end{align*}
\]

with boundary conditions

\[
    u(0, t) = u(L, t) = u_{xx}(0, t) = u_{xx}(L, t) = v(0, t) = v(L, t) = 0.
\]
Suspension Cables

The mass of the cable given by $m_1$ is less than the mass of the roadbed. Dividing across by $m_1$ and $m_2$ gives:

\[ \nu_{tt} - c_1 \nu_{xx} + \delta_1 \nu_t - k_1 (u - v)^+ = \epsilon f(x, t) \quad (17.1) \]

\[ u_{tt} + c_2 u_{xxxx} + \delta_2 u_t + k_2 (u - v)^+ = W_0 \quad (17.2) \]

Boundary conditions:
\[ u(0, t) = u(L, t) = u_{xx}(0, t) = u_{xx}(L, t) = \nu(0, t) = \nu(L, t) = 0. \]

- $\nu$ the distance from the equilibrium of the cable
- $u$ is the displacement of the beam, where both are measured downward
- The stays that connect the beam and the string pull the cable down, which result in the negative signs of equations
- The strings hold the roadbed up, which result in a positive sign for equation
- $c_1$ and $c_2$ are the strengths of the cable and the roadbed.
Suspension Cables

Using no node solutions $u(x, t) = y(t) \sin(\pi x / L)$ and $v(x, t) = z(t) \sin(\pi x / L)$ and $f(x, t) = g(t) \sin(\pi x / L)$ results in:

\[ z'' + \delta_1 z' + a_1 z - k_1 (y - z)^+ = \epsilon g(t) \]  \hspace{1cm} (18.1)

\[ y'' + \delta_2 y' + a_2 y + k_2 (y - z)^+ = W_0 \]  \hspace{1cm} (18.2)

with the same boundary conditions as before.

- $\delta = 0$ and $k_2$ is small, the equations result in small and large periodic solutions
- Galloping cables that move back and forth
- When $a_1 = 10$, $a_2 = .1$, $\delta_1 = \delta_2 = .01$, $k_1 = 10$ and $k_2 = 1$. When $\mu = 4.25$ and $\lambda$ varies from .3 to .4, the bridge will be barely moves
Figure: Nonlinear behavior in the numerical solutions of the simplified cable-bridge equation

(a) \( \mu = 4.25, \; e = 0.4 \)
\( y_1 = 5.0, \; y_3 = -5.0 \)

(b) \( \mu = 4.25, \; e = 0.3 \)
\( y_1 = 5.0, \; y_3 = -5.0 \)
where $\lambda$ increases answers the question of what will happen as the bridge begins to move violently.

- Taking $\mu = 4.5$ and $\lambda = 2.4$, then we obtain a larger motion for the cables that are in an extreme oscillation.
- As $\lambda$ is increased to 3 then the cables are driven by the bridge.
- This suggests why a suspension bridge might behave violently.
- Gusts of winds could cause the cables and towers to move in a high periodic motion which could result in the destruction of the bridge.
Conclusion

Thank you!