Large-Amplitude Periodic Oscillations in Suspension Bridges

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Figure 1: The Golden Gate Bridge
Contents

1 Introduction ................................................................. 3
2 Beginning Model of a Suspension Bridge ......................... 4
3 Partial Differential Equations Model .............................. 6
4 Traveling Waves on the Golden Gate Bridge .................... 8
5 Considering Suspension Cables ..................................... 9
1 Introduction

Suspension bridges are important structures in today’s world. They are capable of spanning very long distances (the longest suspension bridge, the Akashi Kaikyo Bridge, spans a distance of 12,828 feet) and are truly remarkable because they are light and flexible. This causes them to move and vibrate with the presence of wind or other external forces.

The components of the suspension bridge can be divided into two categories; what is above and below the roadway. Below the roadway are large anchors at each end that act as counterweights to the rest of the bridge. There are also piers that cement the towers to the ocean floor. Above the roadway are towers that keep the bridge upright. Connecting the towers are the main cables that run in a concave up pattern across the bridge. Stemming from the main cables are suspender cables that run vertically and connect the main cable to the roadway. A basic diagram of a suspension bridge is included below.

A suspension bridge has multiple forces acting on it. The main forces that act on a suspension bridge are compression and tension. Compression applies balanced force to the pillars of the bridge. Tension is the pulling force that acts on the cables of the bridge. The forces of the suspension bridge act downward, and are distributed throughout the bridge from the towers to the cables and then suspenders and into the anchors. The forces acting on the cables by the deck and the weight of vehicles result in a parabola shape for the main cables.

The suspension bridge has a live load, dead load, and dynamic load. Due to gravity and the weight of the suspension bridge, it can have a tendency to collapse. Live loads consist of vehicles, wind, and changes in temperature and rain. Dead loads consist of the weight of all of the components of the bridge. Dynamic load refers to environmental factors such as earthquakes and wind storms.
2 Beginning Model of a Suspension Bridge

We will begin by introducing a mathematical model for the behavior of the suspension bridge:

\[ u_{tt} + EIu_{xxxx} + \delta u_t = -ku^+ + W(x) + \epsilon f(x, t) \]  

(1)

with boundary conditions \( u(0, t) = u(L, T) = u_{xx}(0, t) = u_{xx}(L, t) = 0 \).

Equation (1) represents a vastly simplified model for the suspension bridge as a beam of length \( L \). Note that we are assuming a model for the suspension bridge with initial conditions equivalent to 0. This will be done throughout our paper. The beam has hinged ends, where \( u(x, t) \) measures the downward deflection of the beam. The second derivative in time, which is the \( u_{tt} \) term, comes from the kinetic energy of the beam (also commonly used in the partial differential wave equation). The \( EIu_{xxxx} \) term comes from the vibrating beam equation that is commonly used in engineering. \( EI \) is a constant parameter, where \( E \) represents the elastic modulus and \( I \) represents the area moment of inertia. Therefore, the product \( EI \) represents the overall stiffness of the beam. We have an additional damping term given by \( \delta u_t \). Also note that we have that \( k \) is the spring constant, and that \( u^+ \) denotes \( u \) when \( u \) is positive and 0 everywhere else. The \( W(x) \) term represents the weight per unit length of the bridge. This is commonly thought of as the weight density function over the length of the bridge. Additionally, we have an external forcing term, modeled by \( \epsilon f(x, t) \). Boundary conditions at both ends of the beam are given.

The model represented in (1) is a partial differential equation model. In the rest of this section, we will simplify it to a model where we can use ordinary differential equations. The partial differential equations model will be explored again later in the paper.

To model the suspension bridge with ordinary differential equations it is necessary to simplify the previous model for the suspension bridge. Here, Lazer and McKenna used separation of variables and simplified the separated solution to only include the first Fourier mode. They replaced:

\( W(x) \) by

\[ W(x) = W_0 \sin(\pi x / L), \]

\( f(x, t) \) by

\[ f(x, t) = f(t) \sin(\pi x / L), \]

and \( u(x, t) \) by

\[ u(x, t) = y(t) \sin(\pi x / L). \]

This results in the simplified model of:

\[ y'' + \delta y' + EI(\pi / L)^4 y + ky^+ = W_0 + \epsilon f(t) \]  

(2)
Now, using the ordinary differential equations model, Lazer and McKenna considered the periodic solutions of the form:

\[ y'' + f(y) = c + g(t), \]
\[ y(0) = y(2\pi), \]
\[ y'(0) = y'(2\pi), \]

with \( f'(\infty) = b \) and \( f'(-\infty) = a \).

We will assume that \( c \) is a constant that is a multiple of the first eigenfunction, and that \( g(t) \) is a periodic function. As the distance between \( a \) and \( b \) increases, then we will have that more and more oscillatory solutions exist, and their order of magnitude is that of \( c \). This means that the ratio between the magnitudes of the solutions will be approximately \( c \).

This implies that the larger the difference between \( a \) and \( b \) and the larger the intervals that the key parameter values lie in, the larger the magnitude of the oscillatory solutions. This can help explain why a suspension bridge could exhibit large wave behavior at an unsafe and unsustainable level. In this case the bridge has to rely more on the spring constant \( k \) and less on the rigidity of the deck, which helps contribute to large magnitude waves traveling through the bridge.
3 Partial Differential Equations Model

We will now return to the partial differential equations model of:

$$u_{tt} + u_{xxxx} + \delta u_t = -ku^+ + W_0 + \epsilon f(x, t)$$

with boundary conditions $u(0, t) = u(L, t) = u_{xx}(0, t) = u_{xx}(L, t) = 0$.

This model is slightly simplified as $EI = 1$ and $W(x) = W_0$ which is a constant. Thus we are assuming that the weight density is constant throughout the bridge.

**Theorem 1** - Let $\delta = 0$, $L = \pi$ and $T = \pi$. In addition, let $f(x, t)$ be an even function in time, $T$-periodic in time and even in $x$ about $\pi/2$. Then if $0 < k < 3$ then (3) has a unique periodic solution of period $\pi$. If $3 < k < 15$, then the equation has a large amplitude periodic solution.

This means that over strengthening the bridge can lead to its destruction because over strengthening the cables may increase the spring constant. This is a big problem because too much tension does not allow the bridge to oscillate. Since the bridge naturally wants to oscillate, it will eventually lead to its collapse.

One interesting result was obtained by setting $\delta = 0.01$, $EI = 1$, $k = 18$, $W = 10$ and by setting the forcing term as $\lambda \sin \mu t \sin \left(\frac{nx}{L}\right)$. Various initial conditions of small or large amplitude were used.

Lazer and McKenna found that with a short bridge, say $L = 3$, the solution would converge to different periodic solutions in finite time, over a large range of $\lambda$ and $\mu$.

However, Lazer and McKenna also found that with a longer bridge, say $L = 6$, that the solution could become unstable and symmetry-breaking would occur.

However, the most interesting numerical result they found was that the solution would almost always converge to what appeared to be a wave, traveling up and down the bridge, and being reflected at the end-points. This can be observed in Figure 3 below.
Figure 3: Multiple solutions of (3) with constant values and a forcing term as described above. Also with $n = 1$ and $\mu$ and $\lambda$ given.
4 Traveling Waves on the Golden Gate Bridge

One specific case that Lazer and McKenna took a look at was the nonlinear phenomena of traveling waves on the Golden Gate Bridge during a violent storm on February 9, 1938. The chief engineer of the bridge reported seeing the “suspended structure of the bridge was undulating vertically in a wavelike motion of considerable amplitude... the oscillations and deflections of the bridge were so pronounced they would seem unbelievable.”

To model solutions of this nature, and to show that their model allowed for traveling waves such as those observed on the Golden Gate Bridge, Lazer and McKenna started with the model given by (3). They assumed there was no external forcing term ($\epsilon f(x, t) = 0$) and no damping term ($\delta = 0$). By normalizing the equation, they took $k = W = 1$. Thus we now have the equation:

$$u_{tt} + u_{xxxx} = -ku^+ + W_0$$

with boundary conditions $u(0, t) = u(L, t) = u_{xx}(0, t) = u_{xx}(L, t) = 0$.

Note that $u \equiv 1$ is an equilibrium solution (this is because if we have $u \equiv 1$, then $u_{tt} + u_{xxxx} = -ku^+ + W_0$ becomes $0 = -(1)(1) + 1 = -1 + 1 = 0$). To simplify to an ODE model, Lazer and McKenna looked for solutions of the form $u(x, t) = 1 + y(x - ct)$. Therefore, $y$ depends on both $x$ and $t$ and satisfies the equation:

$$y^{'''} + c^2y'' + (y + 1)^+ = 1$$

where $y(+\infty) = y(-\infty) = 0$.

Lazer and McKenna solved this ODE by solving the two linear second order equations $y^{'''} + c^2y'' = 1$ for $y < -1$ and $y^{'''} + c^2y'' + y = 0$ for $y \geq -1$ and setting them equal to each other at $y = -1$. The solution can be seen in Figure 4. This Figure is a model of a traveling wave similar to those that was seen at the Golden Gate Bridge.
The next important thing to consider is the motion of the cables in a suspension bridge. The cable will be treated as a vibrating string and will be coupled with the vibrating beam of the roadbed. The roadbed will act as a nonlinear beam with a spring constant given by $k$. There will be no restoring force if the springs are compressed. The model for this system is given by:

$$m_1v_{tt} - Tv_{xx} + \delta_1 v_t - k(u - v)^+ = \epsilon f_1(x, t)$$

(6)

$$m_2u_{tt} + EIu_{xxxx} + \delta_2 u_t + k(u - v)^+ = W_1$$

(7)

with boundary conditions $u(0, t) = u(L, t) = u_{xx}(0, t) = u_{xx}(L, t) = v(0, t) = v(L, t) = 0$.

The mass of the cable given by $m_1$ is less than the mass of the roadbed. Dividing across by $m_1$ and $m_2$ gives:

$$v_{tt} - c_1v_{xx} + \delta_1 v_t - k_1(u - v)^+ = \epsilon f(x, t)$$

(8)

$$u_{tt} + c_2u_{xxxx} + \delta_2 u_t + k_2(u - v)^+ = W_0$$

(9)
with boundary conditions \( u(0, t) = u(L, t) = u_{xx}(0, t) = u_{xx}(L, t) = v(0, t) = v(L, t) = 0 \).

Here \( v \) measures the distance from the equilibrium of the cable and \( u \) is the displacement of the beam, where both are measured downward. The stays that connect the beam and the string pull the cable down, which result in the negative signs of equations (6) and (8). The strings hold the roadbed up, which result in a positive sign for equation (7) and (9). Here \( c_1 \) and \( c_2 \) are the strengths of the cable and the roadbed.

Using no node solutions \( u(x, t) = y(t) \sin(\pi x / L) \) and \( v(x, t) = z(t) \sin(\pi x / L) \) and \( f(x, t) = g(t) \sin(\pi x / L) \) results in:

\[
z'' + \delta_1 z' + a_1 z - k_1 (y - z)^+ = \epsilon g(t) \quad (10)
\]

\[
y'' + \delta_2 y' + a_2 y + k_2 (y - z)^+ = W_0 \quad (11)
\]

with the same boundary conditions as before.

Lazer and McKenna then considered a theoretical result where \( \delta = 0 \) and \( k_2 \) is small. Then we have that equations (10) and (11) result in small and large periodic solutions. As a physical result, the cable has galloping waves that move back and forth. Now we will consider numerical results where \( a_1 = 10, a_2 = .1, \delta_1 = \delta_2 = .01, k_1 = 10 \) and \( k_2 = 1 \). When \( \mu = 4.25 \) and \( \lambda \) varies from .3 to .4, then the bridge will be barely moving.

Now considering the cases where \( \lambda \) increases will answer the question of what will happen as the bridge begins to move violently. Taking \( \mu = 4.5 \) and \( \lambda = 2.4 \), then we obtain a larger motion for the cables that are in an extreme oscillation. As \( \lambda \) is increased to 3 then the cables are driven by the bridge. This suggests why a suspension bridge might behave violently. The gusts of winds could cause the cables and towers to move in a high periodic motion which could result in the destruction of the bridge.
Figure 5: Nonlinear behavior in the numerical solutions of the simplified cable-bridge equation

References
